Centre Number

Candidate Number

ADVANCED
General Certificate of Education

Mathematics

Assessment Unit A2 1

assessing
Pure Mathematics

[AMT11]

PRACTICE PAPER

Summer 2022

TIME
2 hours 30 minutes

INSTRUCTIONS TO CANDIDATES

Write your Centre Number and Candidate Number in the spaces provided at the top of this page.
You must answer **all twelve** questions in the spaces provided.
Do not write outside the boxed area on each page or on blank pages or tracing paper.
Complete in black ink only. **Do not write with a gel pen**.
Questions which require drawing or sketching should be completed using an HB pencil.
Show clearly the full development of your answers. **Answers without working may not gain full credit**.
Answers should be given to three significant figures unless otherwise stated.
You are permitted to use a graphic or scientific calculator in this paper.
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INFORMATION FOR CANDIDATES
The total mark for this paper is 150
Figures in brackets printed down the right-hand side of pages indicate the marks awarded to each question or part question.
A copy of the **Mathematical Formulae and Tables booklet** is provided.
Throughout the paper the logarithmic notation used is ln z where it is noted that $\ln z \equiv \log_e z$

1. **Fig. 1** below shows the graph of

$$y = \ln(2 - x^3)$$

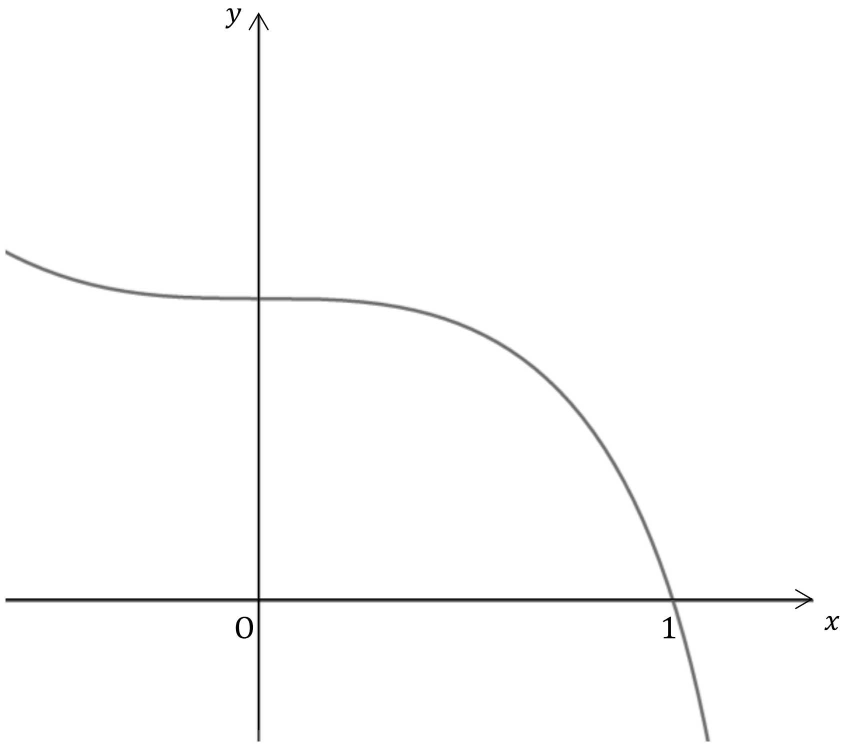

Fig. 1

(i) Using the Trapezium Rule with four strips find an estimate of the area between the curve $y = \ln(2 - x^3)$ and the x-axis from $x = 0$ to $x = 1$ [5]

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(ii) With reference to the graph in **Fig. 1** above, determine whether the answer to part (i) is an over-estimate or an under-estimate of the area. [2]

(iii) Explain how the use of the Trapezium Rule in part (i) could be modified to obtain a better estimate of the area. [1]

2. (a) The first three terms of a Geometric Progression are

$$a, \qquad a + 4, \qquad 2a + 2$$

where $a > 0$

(i) Find the value of a.

[4]

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(ii) Explain why this sequence does not converge. [3]

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(iii) Find the least number of terms required for the sum of the terms in this Geometric Progression to exceed 1520 [5]

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(b) The first four terms in the expansion of

$$(1 + ax)^{-\frac{1}{2}}$$

are $A - 2x + Bx^2 + Cx^3$

(i) Find the values of a, A, B and C. [7]

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(ii) State the values of x for which the equation is valid. [1]

3. **Fig. 2** below shows the graph of the function

$$f(x) = \frac{4x - 5}{x + 2} \qquad x \in \mathbb{R}, \quad x > -2$$

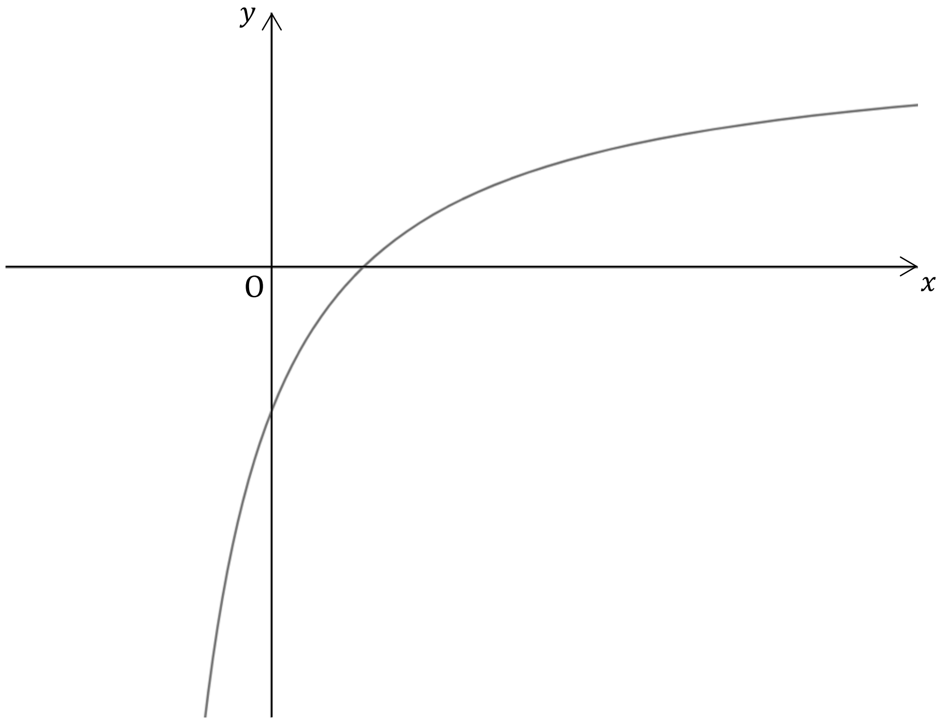

Fig. 2

(i) State the range of $f(x)$. [1]

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(ii) Find the inverse function $f^{-1}(x)$ stating its domain. [5]

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The function $g(x)$ is defined by

$$g(x) = x - 4 \qquad x \in \mathbb{R}$$

(iii) Find the composite function $gf(x)$, stating its domain. [3]

4. (i) Prove the identity

$$\frac{\cot^2\theta + 1}{\cot^2\theta - 1} \equiv \sec 2\theta$$ [6]

(ii) Hence solve the equation

$$\sqrt{3}(\cot^2\theta + 1) = 2(\cot^2\theta - 1) \qquad 0 \le \theta \le \pi$$ [5]

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5. Find the derivative of each of the following expressions:

(i) $(x^2 + 3x + 2)^5$ [3]

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(ii) $\ln x \sin^3 2x$ [5]

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(iii) $\dfrac{x^5}{e^{3x} + 2}$ [4]

6. **Fig. 3** below shows two intersecting circles with centres A and B.

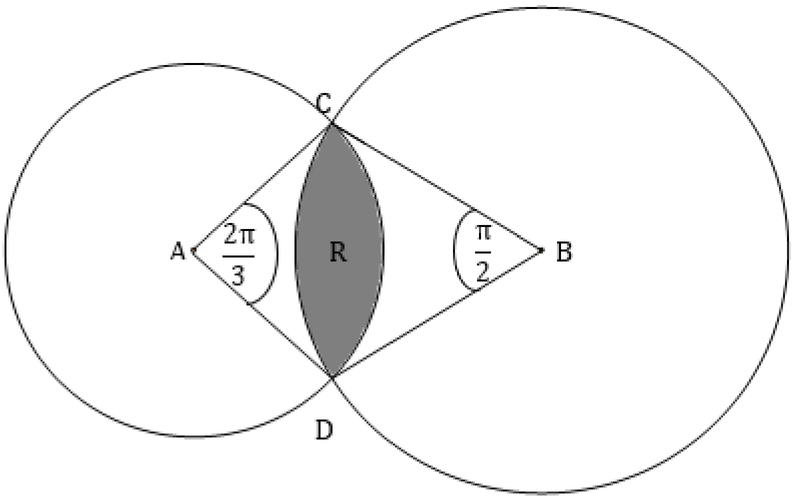

Fig. 3

CD is the common chord of the two circles.

CD = 12 cm.

Angle $\angle CAD = \frac{2\pi}{3}$ radians   Angle $\angle CBD = \frac{\pi}{2}$ radians

Find:

(i) the radius AC; [2]

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(ii) the radius BC; [1]

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(iii) the exact area of the shaded region R. [9]

7. The expression $5 \sin x + 12 \cos x$ can be written in the form $R \sin(x + \alpha)$ where

$R > 0$ and $0 \leq \alpha \leq \frac{\pi}{2}$

(i) Find the values of R and α. [6]

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Fig. 4 below shows the graph of

$$y = \frac{26}{5 \sin x + 12 \cos x} \qquad 0 \leq x \leq 2\pi$$

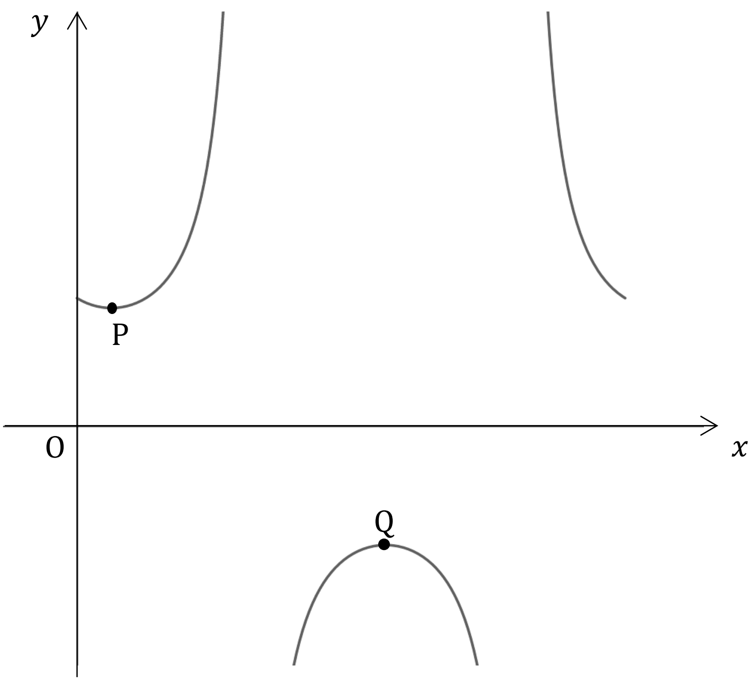

Fig. 4

P and Q are the minimum and maximum turning points respectively on this curve.

(ii) Find the coordinates of P and Q. [5]

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8. Given

$$2x^3 + 3x^2y - 5x = 0$$

use implicit differentiation to show that

$$2(2x + y) + 4x\frac{\mathrm{d}y}{\mathrm{d}x} + x^2\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 0$$ [10]

9. A curve is defined parametrically by

$$x = \cos 2\theta - 2\cos\theta \qquad y = \sin 2\theta + 2\sin\theta$$

(i) Prove that

$$\frac{dy}{dx} = -\frac{1+\cos\theta}{\sin\theta}$$ [8]

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(ii) Show that the equation of the tangent to this curve at the point where $\theta = 0$ is given by $x = a$, clearly stating the value of a. [4]

10. (a) Use the substitution $u = 2x - 1$ to find

$$\int (x+3)(2x-1)^4 \mathrm{d}x$$ [9]

(b) Find

$$\int \sin^2 x \cos^5 x \, \mathrm{d}x$$

[8]

11. Solve the differential equation

$$\frac{dy}{dx} = \frac{y(x-11)}{x^2 - x - 12}$$

given that $y = 1$ when $x = 5$ [13]

12. A logo is to be painted on a rectangular plate as shown in **Fig. 5** below.

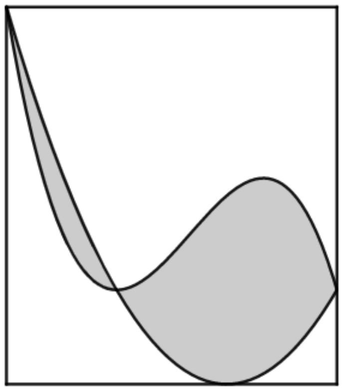

Fig. 5

Fig. 6 below shows the graphs of

$$y = -x^3 + 2x^2 + 1$$

and

$$y = x^2 - 2x + 1$$

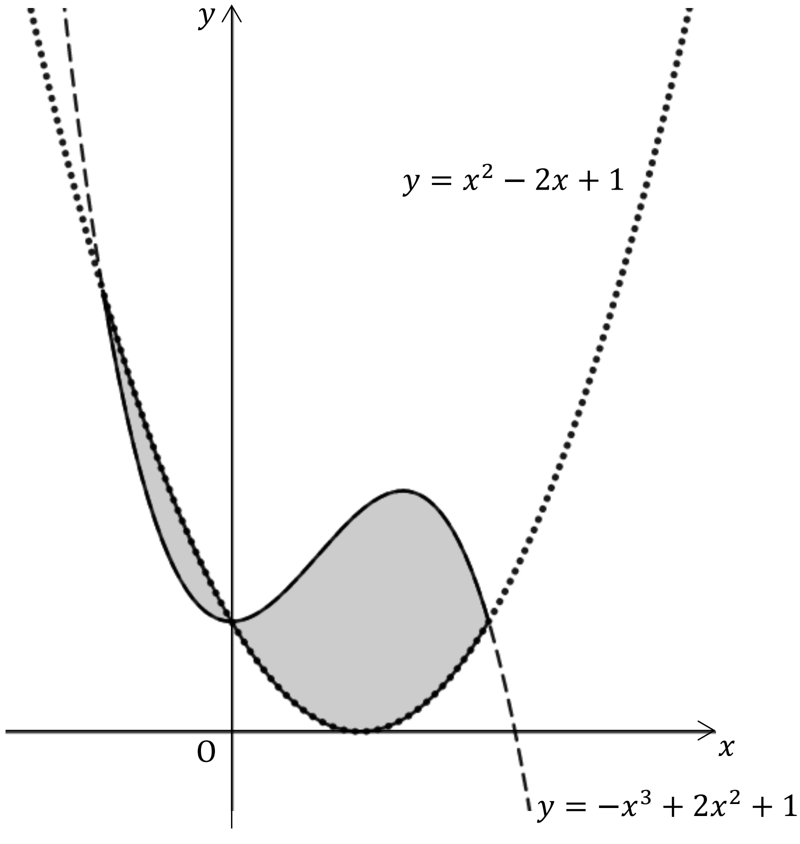

Fig. 6

The painted logo can be modelled as the shape of the shaded regions enclosed between these two graphs.

Find the percentage of the area of the plate which is painted. [15]

THIS IS THE END OF THE QUESTION PAPER

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PRACTICE PAPER
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MARK SCHEME

	AVAILABLE MARKS

1. (i)

x	y
0	$\ln 2$
0.25	$\ln\left(\frac{127}{64}\right)$
0.5	$\ln\left(\frac{15}{8}\right)$
0.75	$\ln\left(\frac{101}{64}\right)$
1	0

MW2
W1

$$\text{Area} = \frac{1}{2}\times\frac{1}{4}\left\{\ln 2 + 2\left(\ln\left(\frac{127}{64}\right)+\ln\left(\frac{15}{8}\right)+\ln\left(\frac{101}{64}\right)\right)\right\}$$ M1

$$= 0.529 \text{ (3sf)}$$ W1

(ii) The upper sloping lines of the individual trapezia lie below the curve. MW1
Therefore the value in (i) in an under - estimate. MW1

(iii) The answer would be more accurate if more strips were used (oe). MW1

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		AVAILABLE MARKS

2. (a)(i) $\frac{a+4}{a} = \frac{2a+2}{a+4}$ M1W1

$(a+4)^2 = a(2a+2)$

$a^2 + 8a + 16 = 2a^2 + 2a$

$a^2 - 6a - 16 = 0$ W1

$(a-8)(a+2) = 0$

$a = 8, \quad a = -2$ (not feasible) MW1

(ii) Common ratio: $r = \frac{a+4}{a} = \frac{12}{8} = 1.5$ M1W1

Since $r > 1$, then the sequence does not converge. MW1

(iii) $a = 8, r = 1.5$

$S_n = \frac{a(r^n - 1)}{r - 1}$

$= \frac{8(1.5^n - 1)}{1.5 - 1}$ M1W1

$\Rightarrow 16(1.5^n - 1) > 1520$ M1

$1.5^n - 1 > 95$

$1.5^n > 96$

$n \ln 1.5 > \ln 96$

$n > 11.257 \ldots$ W1

Hence the least number of terms is 12 W1

(b)(i) $(1+ax)^{-\frac{1}{2}}$

$= 1 - \frac{1}{2}ax + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!}(ax)^2 + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{3!}(ax)^3 + \cdots$ M1W1

$\Rightarrow A = 1$ MW1

$-\frac{1}{2}a = -2$ M1

$a = 4$ W1

$B = \frac{3}{8}a^2$

$= 6$ MW1

$C = -\frac{15}{8 \times 3!}a^3$

$= -20$ MW1

(ii) $|4x| < 1$

$|x| < \frac{1}{4}$ MW1

20

			AVAILABLE MARKS
3.	(i) $x \to \infty \implies \mathrm{f}(x) \to 4$ $\implies \mathrm{f}(x) \in \mathbb{R}, \mathrm{f}(x) < 4$	MW1	
	(ii) $y = \dfrac{4x-5}{x+2}$	M1	
	$\implies y(x+2) = 4x-5$	W1	
	$xy + 2y = 4x - 5$ $x(4-y) = 2y+5$		
	$x = \dfrac{2y+5}{4-y}$	MW1	
	$\implies \mathrm{f}^{-1}(x) = \dfrac{2x+5}{4-x} \quad x \in \mathbb{R}, x < 4$	W2	
	(iii) $\mathrm{gf}(x) = \mathrm{g}\left(\dfrac{4x-5}{x+2}\right)$	M1	
	$= \dfrac{4x-5}{x+2} - 4$	W1	
	$= \dfrac{4x-5-4x-8}{x+2}$		
	$\implies \mathrm{gf}(x) = -\dfrac{13}{x+2} \quad x \in \mathbb{R}, x > -2$	W1	9
4.	(i) $\dfrac{\cot^2\theta + 1}{\cot^2\theta - 1}$		
	$\equiv \dfrac{\dfrac{\cos^2\theta}{\sin^2\theta} + 1}{\dfrac{\cos^2\theta}{\sin^2\theta} - 1}$	M1 W1	
	$\equiv \dfrac{\cos^2\theta + \sin^2\theta}{\cos^2\theta - \sin^2\theta}$	MW1	
	$\equiv \dfrac{1}{\cos 2\theta}$	MW2	
	$\equiv \sec 2\theta$	W1	
	(ii) $\sqrt{3}(\cot^2\theta + 1) = 2(\cot^2\theta - 1)$		
	$\implies \dfrac{\cot^2\theta + 1}{\cot^2\theta - 1} = \dfrac{2}{\sqrt{3}}$	M1	
	$\implies \sec 2\theta = \dfrac{2}{\sqrt{3}}$	MW1	
	$\implies \cos 2\theta = \dfrac{\sqrt{3}}{2}$	W1	
	Related acute angle $= \dfrac{\pi}{6} \qquad 0 \le 2\theta \le 2\pi$		
	$2\theta = \dfrac{\pi}{6}, \quad \dfrac{11\pi}{6}$		
	$\theta = \dfrac{\pi}{12}, \quad \dfrac{11\pi}{12}$	MW2	11

		AVAILABLE MARKS
5. (i) $y = (x^2+3x+2)^5$		
$\frac{dy}{dx} = 5(x^2+3x+2)^4(2x+3)$	M1W2	
(ii) $y = \ln x \sin^3 2x$		
$\frac{dy}{dx} = \ln x \times 3(\sin 2x)^2 \times 2\cos 2x + \frac{1}{x} \times \sin^3 2x$	M1W3	
$= 6\ln x \sin^2 2x \cos 2x + \frac{\sin^3 2x}{x}$	W1	
(iii) $y = \frac{x^5}{e^{3x}+2}$		
$\frac{dy}{dx} = \frac{(e^{3x}+2) \times 5x^4 - x^5 \times 3e^{3x}}{(e^{3x}+2)^2}$	M1W2	
$= \frac{5x^4e^{3x} + 10x^4 - 3x^5e^{3x}}{(e^{3x}+2)^2}$	W1	12
6. (i) $\sin\left(\frac{\pi}{3}\right) = \frac{6}{AC}$	M1	
$AC = \frac{6}{\sin\left(\frac{\pi}{3}\right)}$		
$AC = 4\sqrt{3}$ cm	W1	
(ii) $\sin\left(\frac{\pi}{4}\right) = \frac{6}{BC}$		
$BC = \frac{6}{\sin\left(\frac{\pi}{4}\right)}$		
$BC = 6\sqrt{2}$ cm	MW1	
(iii) Circle centre A		
Area sector $= \frac{1}{2}r^2\theta$		
$= \frac{1}{2} \times (4\sqrt{3})^2 \times \frac{2\pi}{3} = 16\pi$	M1W1	
Area triangle ACD $= \frac{1}{2} \times (4\sqrt{3})^2 \times \sin\left(\frac{2\pi}{3}\right) = 12\sqrt{3}$	M1W1	
$\Rightarrow$ Area segment $= 16\pi - 12\sqrt{3}$	MW1	
Circle centre B		
Area sector $= \frac{1}{2} \times (6\sqrt{2})^2 \times \frac{\pi}{2} = 18\pi$	MW1	
Area triangle BCD $= \frac{1}{2} \times (6\sqrt{2})^2 \times \sin\left(\frac{\pi}{2}\right) = 36$	MW1	
$\Rightarrow$ Area segment $= 18\pi - 36$	MW1	
$\Rightarrow$ Shaded area $= (34\pi - 12\sqrt{3} - 36)$ cm^2	MW1	12

		AVAILABLE MARKS
7. (i) $5\sin x + 12\cos x \equiv R\sin x\cos\alpha + R\cos x\sin\alpha$	MW1	
$\Rightarrow R\sin\alpha = 12$		
$R\cos\alpha = 5$	M1	
$\Rightarrow \tan\alpha = \frac{12}{5}$	M1	
$\alpha = 1.18$ (3sf)	W1	
$R = \sqrt{5^2 + 12^2}$	M1	
$R = 13$	W1	
(ii) $\frac{26}{5\sin x + 12\cos x} \equiv \frac{26}{13\sin(x + 1.176\ldots)}$		
$\equiv \frac{2}{\sin(x + 1.176\ldots)}$	MW1	
P : $\sin(x + 1.176\ldots) = 1$	M1	
$\Rightarrow x = 0.395\,, y = 2$		
$\Rightarrow$ P$(0.395, 2)$	W1	
Q : $\sin(x + 1.176\ldots) = -1$	M1	
$\Rightarrow x = 3.54\,, y = -2$		
$\Rightarrow$ Q$(3.54, -2)$	W1	
		11
8. Differentiate to give	M1W2	
$6x^2 + 3x^2\frac{dy}{dx} + 6xy - 5 = 0$	MW2	
Differentiate a second time to give		
$12x + 6x\frac{dy}{dx} + 3x^2\frac{d^2y}{dx^2} + 6x\frac{dy}{dx} + 6y = 0$	MW3	
$\Rightarrow 12x + 6y + 12x\frac{dy}{dx} + 3x^2\frac{d^2y}{dx^2} = 0$	M1	
$4x + 2y + 4x\frac{dy}{dx} + x^2\frac{d^2y}{dx^2} = 0$		
$2(2x + y) + 4x\frac{dy}{dx} + x^2\frac{d^2y}{dx^2} = 0$	W1	
		10

		AVAILABLE MARKS
9. (i) $\frac{dx}{d\theta} = -2\sin 2\theta + 2\sin\theta$	M1MW1	
$\frac{dy}{d\theta} = 2\cos 2\theta + 2\cos\theta$	MW1	
$\frac{dy}{dx} = \frac{2\cos 2\theta + 2\cos\theta}{-2\sin 2\theta + 2\sin\theta}$	M1 W1	
$= \frac{4\cos^2\theta - 2 + 2\cos\theta}{-4\sin\theta\cos\theta + 2\sin\theta}$	MW2	
$= \frac{2\cos^2\theta + \cos\theta - 1}{-2\sin\theta\cos\theta + \sin\theta}$		
$= \frac{(2\cos\theta - 1)(\cos\theta + 1)}{-\sin\theta\,(2\cos\theta - 1)}$		
$= -\frac{1 + \cos\theta}{\sin\theta}$	MW1	
(ii) $\theta = 0 \implies \frac{dy}{dx} = -\frac{1 + \cos 0}{\sin 0}$	M1	
This is undefined.		
Hence the tangent is a vertical line with equation $x = a$	W1	
$\theta = 0 \implies x = \cos 0 - 2\cos 0$		
$= -1$	MW1	
Therefore the equation of the tangent is $x = -1$	MW1	
		12

	AVAILABLE MARKS

10. (a) $u = 2x - 1$

$\dfrac{du}{dx} = 2$ M1W1

$x = \dfrac{1}{2}(u + 1)$ MW1

$\Rightarrow \int \left(\dfrac{1}{2}(u + 1) + 3\right) u^4 \times \dfrac{1}{2} du$ M1W1

$= \dfrac{1}{2} \int \left(\dfrac{1}{2} u^5 + \dfrac{7}{2} u^4\right) du$ MW1

$= \dfrac{1}{2}\left(\dfrac{1}{12} u^6 + \dfrac{7}{10} u^5\right) + c$ W2

$= \dfrac{(2x - 1)^6}{24} + \dfrac{7(2x - 1)^5}{20} + c$ MW1

(b) $\int \sin^2 x \cos^5 x \, dx$

$= \int \sin^2 x \cos x \cos^4 x \, dx$ M1W1

$= \int \sin^2 x (1 - \sin^2 x)^2 \cos x \, dx$ M1W1

$= \int (\sin^2 x \cos x - 2 \sin^4 x \cos x + \sin^6 x \cos x) dx$ M1W1

$= \dfrac{1}{3} \sin^3 x - \dfrac{2}{5} \sin^5 x + \dfrac{1}{7} \sin^7 x + c$ MW2

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	AVAILABLE MARKS

11. Separate the variables to give

$$\int \frac{\mathrm{d}y}{y} = \int \frac{x-11}{x^2-x-12}\,\mathrm{d}x$$ M1W1

$$\frac{x-11}{(x-4)(x+3)} \equiv \frac{A}{x-4} + \frac{B}{x+3}$$ M1W1

Compare numerators to give

$A(x+3) + B(x-4) \equiv x - 11$ M1

Let $x = 4 \implies 7A = -7$

$A = -1$ MW1

Let $x = -3 \implies -7B = -14$

$B = 2$ MW1

$$\implies \int \frac{\mathrm{d}y}{y} = \int \left(\frac{-1}{x-4} + \frac{2}{x+3}\right)\mathrm{d}x$$

$\implies \ln|y| = -\ln|x-4| + 2\ln|x+3| + c$ M1W2

$x = 5, y = 1 \implies \ln 1 = -\ln 1 + 2\ln 8 + c$

$c = -2\ln 8$ MW1

$$\implies \ln|y| = \ln\left(\frac{|x+3|^2}{64|x-4|}\right)$$ M1

$$y = \frac{(x+3)^2}{64(x-4)}$$ W1

13

AVAILABLE MARKS

12. $-x^3 + 2x^2 + 1 = x^2 - 2x + 1$ M1

$\Longrightarrow x^3 - x^2 - 2x = 0$

$x(x^2 - x - 2) = 0$

$x(x-2)(x+1) = 0$

$x = -1, 0, 2$ W2

Area of 1st Region $= \displaystyle\int_{-1}^{0} \left((x^2 - 2x + 1) - (-x^3 + 2x^2 + 1)\right) \mathrm{d}x$ M1W1

$= \displaystyle\int_{-1}^{0} (x^3 - x^2 - 2x)\, \mathrm{d}x$

$= \left[\frac{1}{4}x^4 - \frac{1}{3}x^3 - x^2\right]_{-1}^{0}$ MW1

$= (0) - \left(\frac{1}{4} + \frac{1}{3} - 1\right)$

$= \frac{5}{12}$ square units M1W1

Area of 2nd Region $= \displaystyle\int_{0}^{2} \left((-x^3 + 2x^2 + 1) - (x^2 - 2x + 1)\right) \mathrm{d}x$ M1

$= \displaystyle\int_{0}^{2} (-x^3 + x^2 + 2x)\, \mathrm{d}x$

$= \left[-\frac{1}{4}x^4 + \frac{1}{3}x^3 + x^2\right]_{0}^{2}$

$= \left(-4 + \frac{8}{3} + 4\right) - (0)$

$= \frac{8}{3}$ square units W1

Total area of logo $= \frac{5}{12} + \frac{8}{3}$

$= \frac{37}{12}$ square units MW1

When $x = -1$, $y = 4$ M1

Base of plate is along x-axis $\Longrightarrow$ Length of rectangle is 3 and height is 4 W1

Area of rectangle = 3×4 = 12 square units MW1

% of area painted $= \dfrac{\left(\frac{37}{12}\right)}{12} \times 100$

$= 25.7\%$ (3sf) MW1 | 15

150