FACTFILE: GCE PHYSICS AS

BRIDGING MATHEMATICS

Rewarding Learning

The following booklet is for teachers to use with students as required; its purpose is to bridge the gap between GCSE Mathematics and A level Physics.

Topics

1. Expressing units in Physics
2. Scientific notation (prefixes and multiples)
3. Converting units
4. Rearranging algebra (including mapping)
5. Graphs
6. Degrees and Radians
7. Trigonometry
8. Vectors
9. Logarithms
10. Exponentials

Expressing units in Physics

Here are some units you will be familiar with from GCSE Physics:

velocity is measured in m/s
acceleration is measured in m/s^2
density is measured in kg/m^3
frequency is measured in Hz

However at AS and A2 level, equations will start to become more complex and so the units used to express quantities will also be more complex.
For example, here is an equation used to express the force experienced due to gravity (called Newton's law of gravitation)

$$F = \frac{GM_1M_2}{r^2}$$

To find the units for G, first we would have to rearrange the equation to make G the subject

$$G = \frac{Fr^2}{M_1M_2}$$

Then work out the units of G = Nm^2/kg^2

But what about finding units for C (the specific heat capacity) from the following equation

$$Q = mc\Delta\theta$$

Where Q is energy measured in Joules, m is mass measured in kg and $\Delta\theta$ is temperature change measured in kelvin.

Again, rearranging the equation we get

$$c = \frac{Q}{m\Delta\theta}$$

So the units of c are J/(kg K)

These are not easy units to interpret and so a new way must be considered to express units. We will adopt the rules for expressing indices in mathematics.

Indices notation

$$\frac{1}{x} = x^{-1} \ , \ \frac{1}{x^2} = x^{-2} \ , \ \sqrt{x} = x^{½} , \ \frac{1}{\sqrt{x}} = x^{-½} \ , \ \sqrt[3]{x} =$$

$$x^{⅓}, \ \frac{1}{\sqrt[3]{x}} = x^{-⅓} \ , \sqrt[3]{x^2} = x^{⅔}, \frac{1}{\sqrt[3]{x^2}} = x^{-⅔}$$

Let's apply this notation to units
When we divide units we express that unit to the power of -1, see the following examples:

m/s = m s^{-1}, m/s^2 = m s^{-2}, kg/m^3 = kg m^{-3}, Hertz = s^{-1},

Nm^2/kg^2 = N m^2 kg^{-2} and J/(kg K) = J kg^{-1} K^{-1}

Try correctly expressing the units for the following quantities:

a. Momentum (Momentum, p = mv)
b. Resistance (R = $\frac{V}{I}$)
c. Power (P = $\frac{E}{t}$)
d. Impulse (Impulse = Ft)
e. Pressure (P = $\frac{F}{A}$)
f. Force (F = ma)
g. Energy (KE = ½mv^2)
h. Charge (Q = It)
i. Volt (V = $\frac{E}{Q}$)

Answers

(a) kg ms^{-1}, (b) VA^{-1}, (c) Js^{-1}, (d) Ns, (e)Nm^{-2}, (f) kg ms^{-2}, (g) kg m^2s^{-2} , (h) As, (i) JC^{-1}

Scientific Notation (prefixes and multiples)

Expressing numbers in the form x^n is a common mathematical operation, for example:

$3 \times 3 \times 3 \times 3 = 3^4$ where 3 is the base and 4 is the index

Powers of ten can be used to simplify very large or very small numbers:

The speed of light, c = 300 000 000 m s^{-1} should be written as $c = 3 \times 10^8$ m s^{-1}

Avogadro's constant N_A = 602 300 000 000 000 000 000 000 should be written as $N_A = 6.023 \times 10^{23}$

Expressing numbers using scientific notation saves writing a long string of zeros and means you are less likely to make a mistake. We can also use this notation to express very small numbers, for example:

The Universal Gravitational constant G = 0.000 000 000 066 7 N m^2 kg^{-2} should be written as 6.67×10^{-11} N m^2 kg^{-2}
The unit of elementary charge e = 0.000 000 000 000 000 000 16 C should be written as $e = 1.6 \times 10^{-19}$ C
This is referred to as standard form, a number between 1 and 10 times a power.

Prefixes
In Physics, we often use prefixes to describe the size of a quantity, for example:

A power of 8 000 W can be written as 8×10^3 W = 8 kW
A current of 0.003 A can be written as 3×10^{-3} A = 3 mA

The following table contains the prefixes that will be important in your A level studies.

submultiple	Prefix (symbol)	multiple	Prefix (symbol)
10^{-2}	centi (c)		
10^{-3}	milli (m)	10^{3}	kilo (k)
10^{-6}	micro (μ)	10^{6}	mega (M)
10^{-9}	nano (n)	10^{9}	giga (G)
10^{-12}	pico (p)	10^{12}	tera (T)
10^{-15}	femto (f)		

Practice Questions

1. Write down the following quantities in standard form without any prefixes.
   a) 470 pm
   b) 1.5 kV
   c) 50 MW
   d) 45 ns

2. All physical quantities consist of a magnitude and a unit, express the following quantities below using the alternative unit indicated in brackets. The first has been completed as an example.
   a) 2.42 m (cm) …….. 2.42×10^{2} cm
   b) 863 μF (F)
   c) 7.34×10^{5} V (kV)
   d) 4.82×10^{-7} MJ (mJ)

3. Calculate the following and express the result as a number and a unit with no prefix.

   a) $\frac{470\ pJ}{1.5\ ks}$ b) $\frac{50\ MJ}{40\ ns}$ c) $\frac{500\ mm}{8.5\ \mu s}$ d) $\frac{40\ MJ}{1.5\ cm}$

Answers

1 (a) 4.7×10^{-10} m (b) 1.5×10^{3} V (c) 5.0×10^{7} W (d) 4.5×10^{-8} s

2 (b) 8.63×10^{-4} F (c) 734 kV (d) 482 mJ

3 (a) 3.13×10^{-13} J s^{-1} (b) 1.25×10^{15} W (c) 5.88×10^{4} m s^{-1} (d) 2.67×10^{9} N

Converting units

In Physics we often have to change from one unit to another. Here are some common examples and methods for changing units.

Speed

The speed of a car can be changed from kmh^{-1} to $m\ s^{-1}$ as follows:

70 km per hour is 70 x 1 000 metres in one hour

This is 70 000 metres in 3 600 seconds which is 19.4 $m\ s^{-1}$

So the method for changing kmh^{-1} to ms^{-1} is x $\frac{1000}{3600}$

Density

The density of a material can be changed from $kg\ m^{-3}$ into $g\ cm^{-3}$ as follows:

The density of aluminium is 2700 kilograms per metre cubed

This is 2 700 000 grams per metre cubed

1 m^3 = 100 x 100 x 100 cm^3 = 1 000 000 cm^3

So 2 700 000 grams per 1 000 000 cm^3 = 2.7 $g\ cm^{-3}$

Area

The area of cross section of a wire can be changed from mm^2 to m^2 as follows:

The diameter of a wire is 0.4 mm, this is 0.4 x 10^{-3} m

The equation for area is $\pi r^2 = \frac{\pi d^2}{4}$ where r is the radius and d is the diameter of the wire.

So π x $\frac{(0.4\ x\ 10^{-3})^2}{4}$= 1.26 x 10^{-7} m^2 = 0.000 000 126 m^2

If this calculation was completed in cm^2 the answer would be

π x $\frac{(0.4\ x\ 10^{-1})^2}{4}$ = 1.26 x 10^{-3} cm^2 = 0.001 26 cm^2

If this calculation was completed in mm^2 the answer would be

π x $\frac{0.4^2}{4}$ = 0.126 mm^2

So 0.126 mm^2 = 0.126 x 10^{-2} cm^2 = 0.126 x 10^{-6} m^2

So you can see to change from mm^2 to cm^2 to m^2 you need to multiply by the conversion factor squared:

The conversion factor to change from mm to cm is $\frac{1}{10}$ = 10^{-1}

And so to change from mm^2 to cm^2 the conversion factor is $\frac{1}{10^2}$ = 10^{-2}

The conversion factor to change from mm to m is $\frac{1}{1000}$ = 10^{-3}

And so the conversion factor to change from mm^2 to m^2 = $\frac{1}{1000^2}$ = 10^{-6}

Volume

This same principle applies when changing mm^3 or cm^3 to m^3 when expressing volumes.

A volume of 15 mm^3 = 15 x $\frac{1}{1000^3}$ = 15 x 10^{-9} m^3 (we use cubed instead of squared this time as volume is not expressed in m^2 but in m^3)

A volume of 25 cm^3 = 25 x $\frac{1}{100^3}$ = 25 x 10^{-6} m^3

The volumes of liquids are often expressed in ml and litres,
where 1000 ml = 1 litre. Now since 1 ml = 1 cm^3, 1 litre = 1000 cm^3
there are 1 x 10^6 cm^3 in 1 m^3 and there will be 1 x 10^3 litres in 1 m^3

Practice Questions

1. Convert the following quantities into the units stated in brackets

   (a) 2.34 mm (m), (b) 8.44 cm^3 (m^3), (c) 0.53 mm^3 (m^3)

   (d) 75 ml (m^3), (e) 23 l (m^3), (f) 0.55 mm^2 (m^2)
2. Which is longer, 3 kiloseconds or 1 microcentury?
3. Express the following densities in kg m^{-3}

   (a) 21.5 g cm^{-3} (platinum), (b) 7.8 g cm^{-3} (steel), (c) 0.0009 g cm^{-3} (air)
4. Express the following speeds in ms^{-1}

   (a) 100 km h^{-1}, (b) 78 km h^{-1}, (c) 120 km h^{-1}
5. Normal human blood contains 5.1 x 10^6 red blood cells per millimetre cubed. The total volume of blood in a 60 kg woman is 5 litres. How many red blood cells does this person have?

Answers

1 (a) 2.34 x 10^{-3} m, (b) 8.44 x 10^{-6} m^3, (c) 5.3 x 10^{-10} m^3, (d) 7.5 x 10^{-5} m^3,

(e) 2.3 x 10^{-2} m^3, (f) 5.5 x 10^{-7} m^2

2 3 kiloseconds = 50.0 minutes whereas 1 microcentury = 52.56 minutes

3 (a) 21500 kg m^{-3}, (b) 7800 kg m^{-3}, (c) 0.9 kg m^{-3}

4 (a) 27.7 m s^{-1}, (b) 21.7 m s^{-1}, (c) 33.3 m s^{-1}

5 5 litres = 5 x 1000 ml = 5000 cm^3 = 5 x 10^6 mm^3

5 x 10^6 x 5.1 x 10^6 = 2.55 x 10^{13} red blood cells

Rearranging Algebra / Changing the subject of an equation

You should already be comfortable rearranging algebraic equations, so this section is designed to give you practice rearranging Physics equations. Here is an example you should already by familiar with:

The equation $v = u + at$ can be rearranged to make *a* or *t* the subject

$$a = \frac{v-u}{t} \text{ and } t = \frac{v-u}{a}$$

Practice Questions

1. Here are some other equations; can you rearrange then to make the letter in brackets the subject?

    a. $v^2 = u^2 + 2as$ (a)
    b. $P = I^2 R$ (I)
    c. $P = \frac{V^2}{R}$ (R)
    d. $T = 2\pi\sqrt{\frac{l}{g}}$ (l)
    e. $V = E - Ir$ (r)
    f. $V = \frac{4}{3}\pi r^3$ (r)

2. Write down the simplest expression for the density ρ of a sphere in terms of its mass *m* and the diameter *d*. The volume of a sphere of radius *r* is $\frac{4}{3}\pi r^3$.

3. A helical spring has mass *m* attached to one end. This produces a force *F* in the spring. The mass is then displaced and released causing it to oscillate. The equation below represents the relationship for the periodic time (time for one oscillation) *T* of a mass-spring system.

$$T = 2\pi\sqrt{\frac{mx}{F}}$$

Rearrange the equation to make x the subject and find its units in terms of m, s and kg.

4. The radius *R* of an atomic fireball depends on the energy *E*, the density *ρ* of the air and the time *t* after detonation.

   The equation is

$$R^5 = \frac{kt^2E}{\rho}$$

   Rearrange the equation to make *k* the subject and show that *k* has no units.

5. The equation describing the flow of current in a wire *I* is given by the equation $I = nAve$ where *n* is the number of electrons per m^3 of wire, *A* is the cross sectional area of the wire, *v* is the drift velocity in the wire and *e* is the charge carried by an electron. Rearrange the equation to make *v* the subject and show that both sides of the equation have units $m\ s^{-1}$.

Answers

1 (a) $a = \frac{v^2 - u^2}{2s}$, (b) $I = \sqrt{\frac{P}{R}}$, (c) $R = \frac{V^2}{P}$, (d) $T^2 = 4\pi^2 \frac{l}{g}$, so $l = \frac{T^2 g}{4\pi^2}$

(e) $r = \frac{E - V}{I}$, (f) $r = \sqrt[3]{\frac{3V}{4\pi}}$

2 $\rho = \frac{6m}{\pi d^3}$

3 $x = \frac{T^2 F}{4\pi^2 m}$, units of x are metres (m)

4 $k = \frac{R^5 \rho}{t^2 E}$, k has no units because numerator and denominator both have units kg m^2

5 $v = \frac{I}{nAe}$, RHS has units A $m^{+3}m^{-2}$ $(A\ s)^{-1}$ = ms^{-1} and LHS has units of velocity (m s^{-1})

Graphs/Mapping equations onto the equation of a straight line

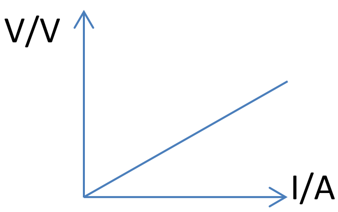
$y = mx + c$

Any equation can be plotted as a straight line plot, provided there is one independent and one dependant variable (the independent variable is the variable you change in an experiment and the dependent variable is the one you take measurements of during the experiment). All other quantities must be constant or controlled.

For example, an experiment you tried at GCSE was finding the resistance of a wire at **constant temperature**. By increasing current in steps of say 0.5 A and measuring the voltage across the resistor, a set of readings were obtained. To map the equation V = IR onto the equation for a straight line
y = mx + c , follow the example laid out below.

$$V = R\ I$$
$$\downarrow \quad \downarrow \quad \downarrow$$
$$y = m\ x + C$$

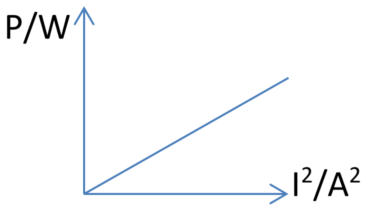
So V is plotted on the Y axis and I on the X axis, R is the gradient and there is no Y intercept, so the graph will go through the origin.



What about the equation $P = I^2R$ where R is the constant?
Rearrange the equation into the format y = mx + c

$$P = R\ I^2$$
$$\downarrow \quad \downarrow \quad \downarrow$$
$$y = m\ x + c$$

So P is plotted on the Y axis, I^2 on the X axis, R is the gradient and there is no Y intercept, so the graph will go through the origin.



Practice Questions

Can you identify the quantity to be plotted on the Y axis and X axis? What will the gradient be and is there a Y intercept?

(a) $W = ½kx^2$ *where k is the constant*

(b) $F = \frac{GMm}{r^2}$ *where G, M and m are constants*

(c) $C = \frac{\varepsilon_r \varepsilon A}{d}$ *where d and ε are constants*

(d) $C = \frac{\varepsilon A}{d}$ *where A and ε are constants*

(e) $E = \frac{V}{d}$ *where V is constant*

(f) $c = f\lambda$ *where c is constant*

(g) $T = 2\pi\sqrt{\frac{l}{g}}$ *where g is constant*

(h) $T = 2\pi\sqrt{\frac{m}{k}}$ *where k is constant*

(i) $eV = hf - \phi$ *where e, h and* ϕ *are constants*

(j) $E = V + Ir$ *where r and E are constants*

Answers

(a) W vs x^2 will be a straight line graph through the origin and the gradient will be $\frac{k}{2}$

(b) F vs $\frac{1}{r^2}$ will be a straight line graph through the origin and the gradient will be GMm

(c) C vs A will be a straight line graph through the origin and the gradient will be $\frac{\varepsilon}{d}$

(d) C vs $\frac{1}{d}$ will be a straight line graph through the origin and the gradient will be εA

(e) E vs $\frac{1}{d}$ will be a straight line graph through the origin and the gradient will be V

(f) f vs $\frac{1}{\lambda}$ will be a straight line graph through the origin and the gradient will be c

(g) T^2 vs l will be a straight line graph through the origin and the gradient will be $\frac{4\pi^2}{g}$

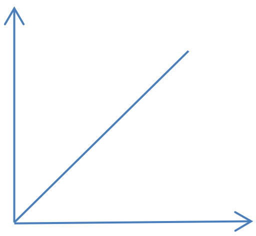
(h) T^2 vs m will be a straight line graph through the origin and the gradient will be $\frac{4\pi^2}{k}$

(i) V vs f will be a straight line with the gradient equal to $\frac{h}{e}$ and Y intercept = -$\frac{-\phi}{E}$

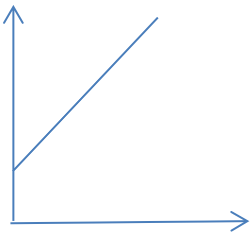
(j) V vs I will be a straight line with the gradient equal to –r and Y intercept = E

In Physics you will often be asked to sketch graphs to demonstrate relationships. Here are important graph sketches that you will need to know at A level.

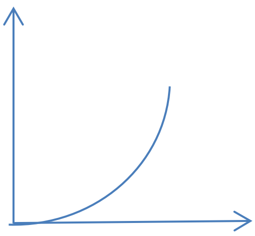
y = mx



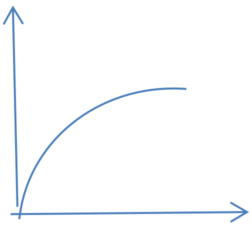
y = mx + c



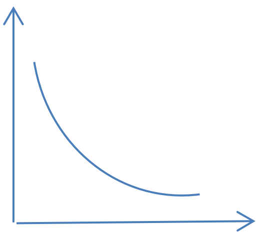
$y = kx^2$



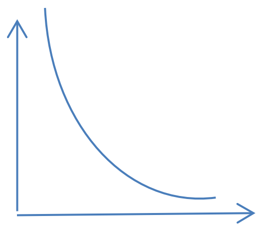
$y = kx^{½}$



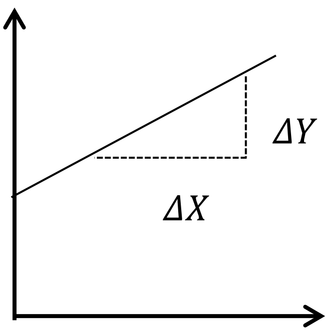
$y = \frac{k}{x}$



$y = \frac{k}{x^2}$



We need to examine the y = mx + c graph more carefully.



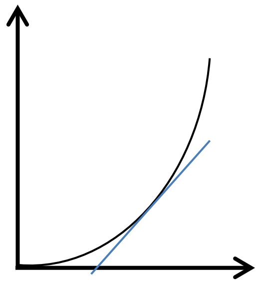
The gradient of this graph $k = \frac{\Delta Y}{\Delta X}$

To do this chose any two points on the line and form a right angled triangle. From this section of the straight line, the change in Y, ΔY is given by $Y_2 - Y_1$ and similarly $\Delta X = X_2 - X_1$

The units of the constant k are given by $\frac{units\ of\ Y}{units\ of\ X}$ which are written as units of Y (units of X)$^{-1}$.

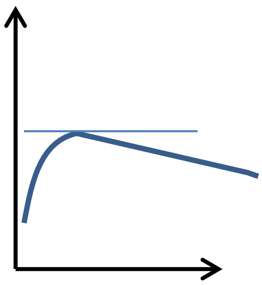
Always choose a large triangle to reduce the error in calculating the gradient.

For graphs which are non linear



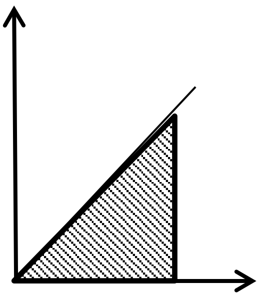
This is a $y = kx^2$ graph. The gradient of this graph is getting steeper. We can see this as the line is curving up. To find the gradient at any point you need to draw a tangent to the curve at that point and then calculate k in the usual way by finding the gradient of the straight line.

Maximum or minimum turning points



At a turning point, the gradient of the curve is zero. This can be seen on the graph as a tangent which is horizontal. Since $\Delta Y = 0$, the gradient = 0

The area under a graph



The area under* a graph can often represent a physical quantity

- The area under a force vs distance graph represents work done
- The area under a power vs time graph represents energy
- The area under a force vs time graph represents impulse
- The area under a potential difference vs charge graph is energy
- The area under a velocity vs time graph represents displacement

* by ‘under’ we mean: between the plot and the horizontal axis.

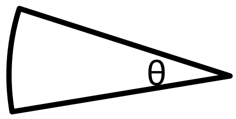
Degrees and Radians

Angles can be measured in degrees and radians. Angles measured in degrees are written with units ° e.g. 60°. Angles in radians are written with units rad e.g. 2.6 rad. To change from degrees to radians or from radians to degrees you will need the following conversion:

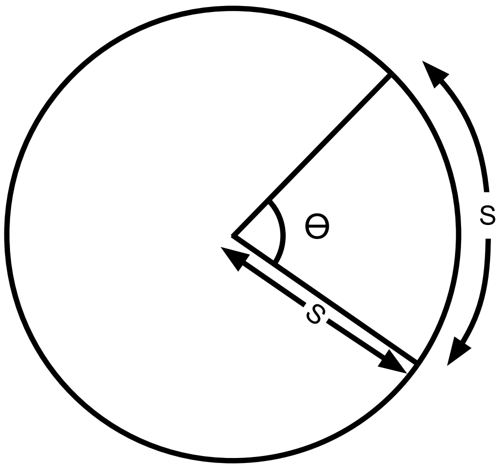
$$2\pi \text{ rad} \equiv 360^\circ$$
$$2 \times 3.14 \text{ rad} \equiv 360^\circ$$
$$6.28 \text{ rad} \equiv 360^\circ$$
$$1 \text{ rad} \equiv 57.3^\circ$$

What is 1 degree?

If a circle is cut into **360 identical** pieces then 1 degree will be the angle of one of those pieces as shown below.



What is a radian?



$\theta^r = \frac{s}{r}$ the ratio of the arc length to the radius

when s = circumference = $2\pi r$ $\theta^r = \frac{2\pi r}{r} = 2\pi$

So $360^\circ \equiv 2\pi$ rad

And 1 radian is equivalent to 57.3°

Practice Questions

1 Express the following angles in degrees

(a) 2π rad (b) π rad (c) $\frac{\pi}{2}$ rad (d) $\frac{\pi}{3}$ rad (e) $\frac{\pi}{6}$ rad (f) 3π rad

2 Express the following angles in degrees

(a) 1 rad (b) 2 rad (c) 0.1 rad (d) 4.25 rad (e) 2.18 rad (f) 0.25 rad

3 Express the following angles in radians as a fraction of 2π or multiple of π

(a) 90° (b) 60° (c) 30° (d) 180° (e) 360° (f) 120°

(g) 240° (h) 225° (i) 720°

Answers

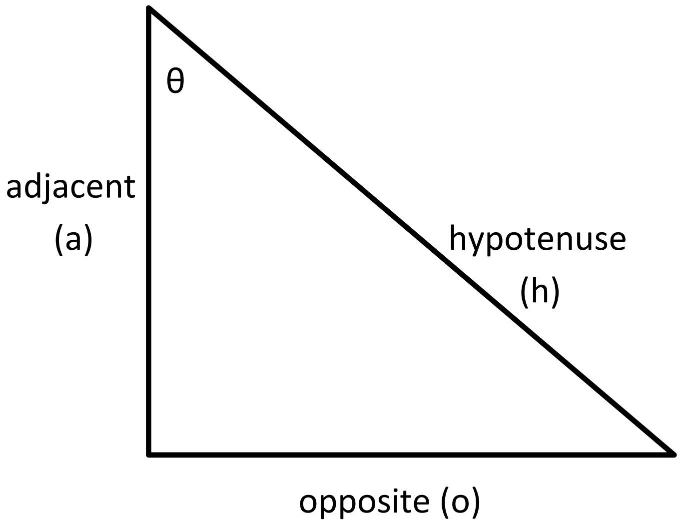
1 (a) 360° (b) 180° (c) 90° (d) 60° (e) 30° (f) 540°

2 (a) 57.3° (b) 115° (c) 5.73° (d) 244° (e) 125° (f) 14.3°

3 (a) $\frac{\pi}{2}$ rad (b) $\frac{\pi}{3}$ rad (c) $\frac{\pi}{6}$ rad (d) π rad (e) 2π rad (f) $\frac{2\pi}{3}$ rad (g) $\frac{4\pi}{3}$ rad (h) $\frac{5\pi}{4}$ rad (i) 4π rad

Trigonometry

Here are the trigonometry functions for right angled triangles everyone should know from GCSE Mathematics.



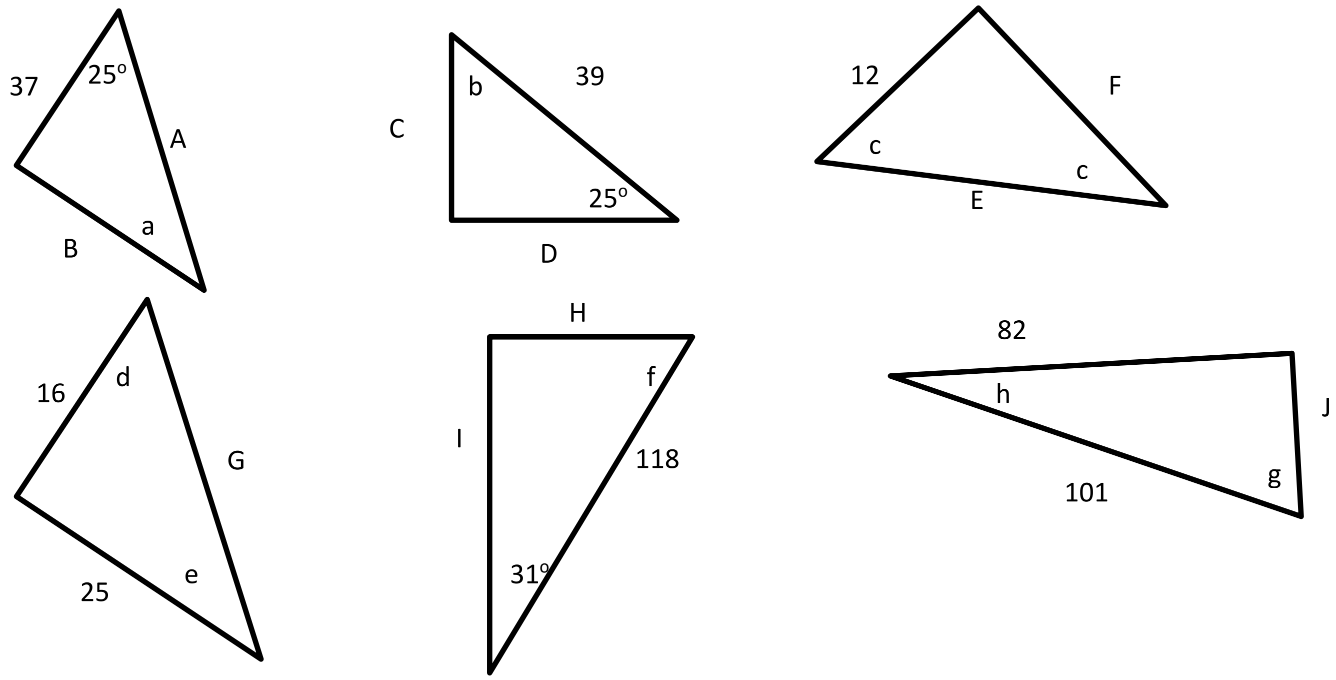
$\sin\theta = \frac{o}{h}$ $\cos\theta = \frac{a}{h}$ $\tan\theta = \frac{o}{a}$

Pythagoras equation

$$h^2 = o^2 + a^2$$

Practice Questions

Find the missing sides and angles in the following right angled triangles



Answers

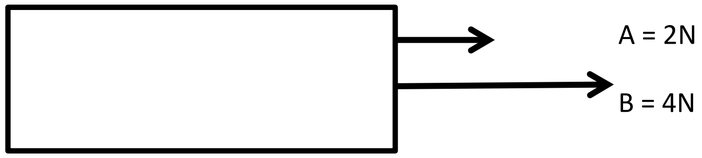
Answers

A = 40.8, B = 17.3, C = 16.5, D = 35.3, E = 17.0, F = 12, G = 29.7, H = 60.8, I = 101, J = 59.0

a = 65°, b = 65°, c = 45°, d = 47°, e = 33°, f = 59°, g = 54°, h = 36°

Vectors

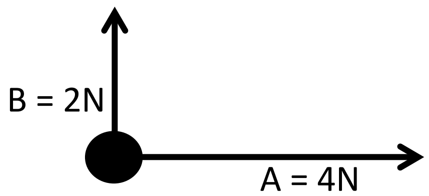
When considering any quantity which is a vector, direction must be taken into account. Examples of vectors you have encountered in GCSE Physics are force, displacement, velocity, momentum and acceleration.
Consider the addition of the following forces:



The resultant force acting is 6 N to the right. (Note the size of the arrow on the diagram represents the magnitude of the force)

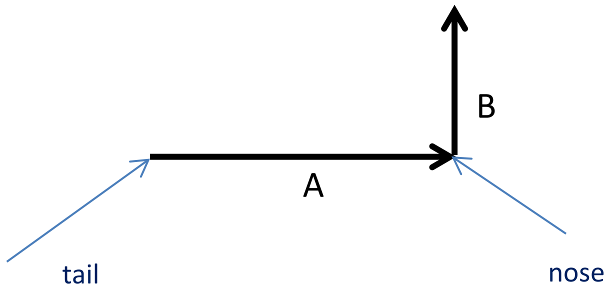
The resultant force acting is 2 N to the left.

However, considering the following



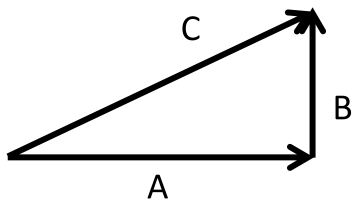
Adding vectors which are not parallel

Firstly we need to know the rule for adding vectors; **when adding vectors we add them nose to tail.**



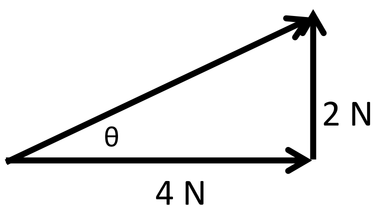
This is the vector addition A + B

We can represent the vectors A + B with a single vector C which could replace both of them i.e. a single force that on its own would provide a force of the same magnitude and in the same direction. This is called the **resultant vector**.



C = A + B

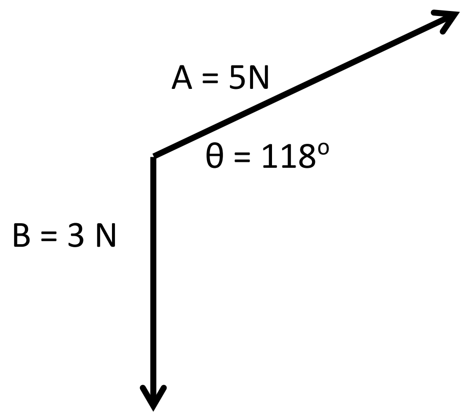
We cannot add or subtract A and B directly since they are not in the same direction, but we can however use our work on trigonometry to find C.



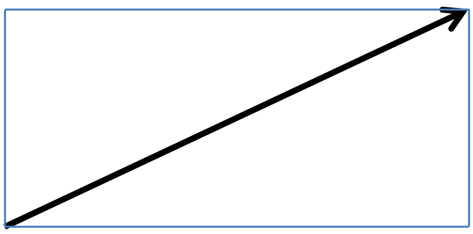
$C = \sqrt{(4^2 + 2^2)} = 4.5$

Since vector C cannot be described without a direction, we need also to find θ. Tan $\theta = \frac{2}{4}$ and so $\theta = 27^o$. So the resultant vector C is 4.5 N at an angle of 27^o to the horizontal.

Resolving vectors

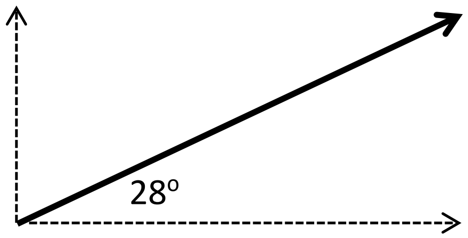


We could **replace vector A with only horizontal and vertical components** so when we added them to B we would always make a right angle triangle. **This is called resolving.**



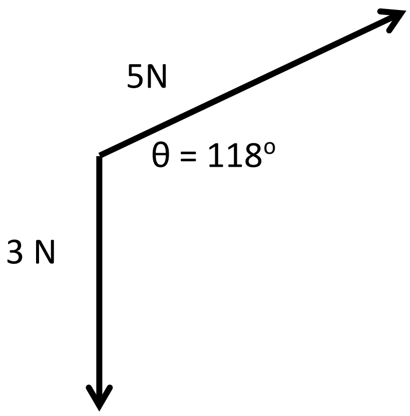
As you can see vector A can made be into the diagonal of a rectangle.

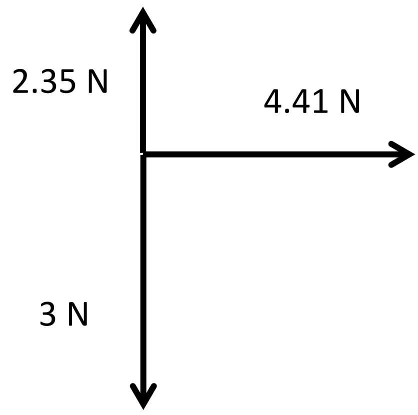
We can split A into horizontal and vertical components as shown below



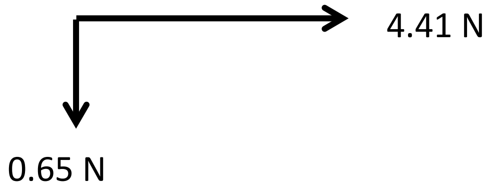
The vertical side of the triangle can be calculated from sin 28 = $\frac{Y}{5}$ and so Y = 5 sin28 = 2.35 and the horizontal side of the triangle can be calculated from cos 28 = $\frac{x}{5}$ and so x = 5 cos28 = 4.41.

Therefore we can redraw the diagram ……. like this

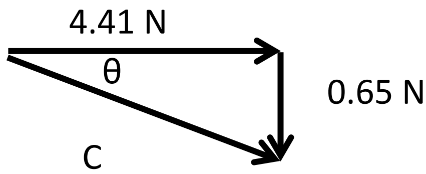




Now we can subtract the vertical vectors to give 3 – 2.35 = 0.65 N down.



Now add the vectors nose to tail and use trigonometry to find C.

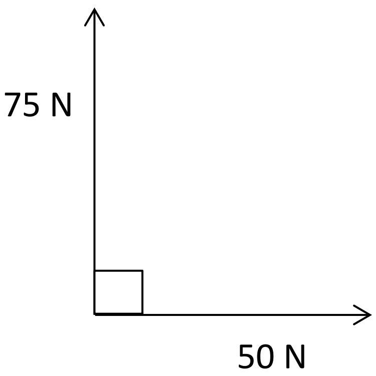


$C = \sqrt{(4.41^2 + 0.65^2)} = 4.46$ N and the angle C makes with the horizontal is, as before, $\tan\theta = \frac{0.65}{4.41} = 8.38^o$

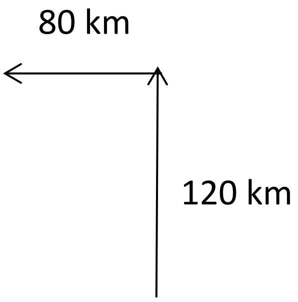
Practice Questions

In questions 1 – 5, find the single vector (resultant vector) which would replace the vectors given. Also provide the angle that this vector would make with the horizontal or vertical.

1

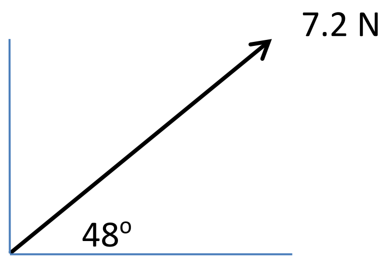


2



In questions 3 – 6, resolve the vectors and calculate the horizontal and vertical components.

3



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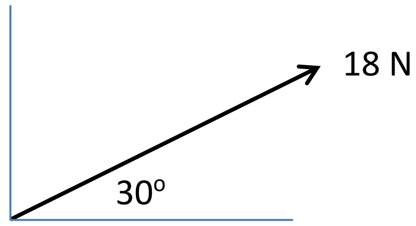
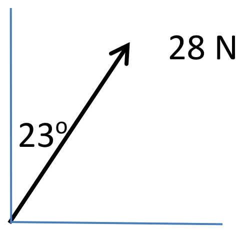
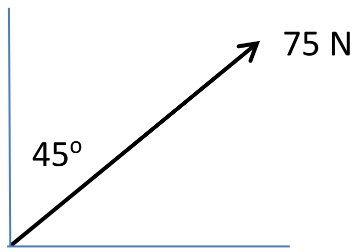


5



6



Answers

1) 90.1N, 33.7° to vertical

2) 144 km, 33.7° to vertical

3) horizontal component = 4.82 N, vertical component = 5.35 N

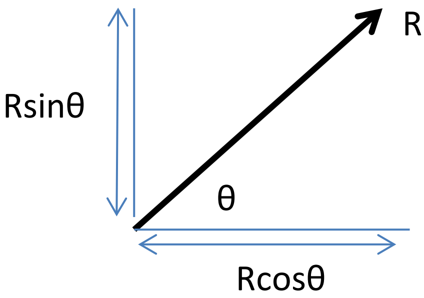
4) horizontal component = 15.6 N, vertical component = 9 N

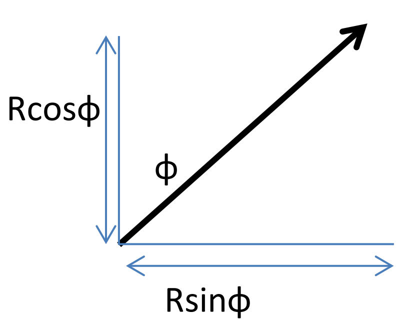
5) horizontal component = 10.9 N, vertical component = 25.8 N

6) horizontal component = 53.0 N, vertical component = 53.0 N

Points to note:

When finding horizontal and vertical components of a vector you should have observed by now





A few trigonometry rules that might come in useful:

$\sin 90^\circ = 1$ $\qquad$ $\cos 90^\circ = 0$

$\sin 0^\circ = 0$ $\qquad$ $\cos 0^\circ = 1$

$\sin 45^\circ = \cos 45^\circ$

$\sin \theta = \cos (90^\circ - \theta)$

$\tan \theta = \frac{\sin \theta}{\cos \theta}$

$\sin 2\theta = 2 \sin\theta \cos\theta$

Exponentials and Log

In Physics we often work with powers such as squared and cubed e.g.

Find x^2 when $x = 4.5$

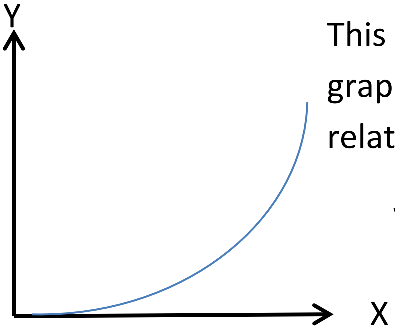
$4.5^2 = 4.5 \times 4.5 = 20.25$

We also work with fractional powers e.g.

Find √x i.e. $x^{½}$ when $x = 17$

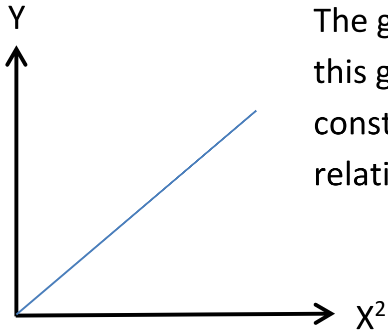
$17^{½} = 4.12$

Many of the experiments we investigate will have results where Y is proportional to x^2 or some other power. We can plot these graphs as follows:



This is the graph of the relationship

$y = kx^2$



The gradient of this graph, k, is the constant in the relationship

$y = kx^2$

A directly proportional plot allows us to find the constant k using a wide range of experimental results. However, what if the power is unknown e.g. $y = kx^n$ where n is the unknown power. As you can imagine, it would be very time consuming guessing n and then trying to plot each guess as a directly proportional plot. A way to tackle this problem is to use logarithms.

What are logarithms?

When we take an expression such as 2^3 we use the y^x or x^y operation (depending on your model of calculator) on the calculator to find the answer.

$2^3 = 8$

But what about the following

$2^n = 16$

The inverse operation would be as follows

$\log_2 16 = 4$

The logarithm of a number is the power to which you must raise a base in order to obtain the number. In this example 16 is the number, 2 is the base and 4 is the power,

$$\text{So if } y = x^n \quad \text{then } \log_x y = n$$

Log to the base 10

One of the base numbers we work with in Physics is base 10. Log to the base 10, written $\log_{10}$ or lg is [log] on the calculator.

Try the following on your calculator $10^7 = 10\ 000\ 000$

So $\log 10\ 000\ 000 = \log 10^7 = 7$

$10^3 = 1000$

$\log 1\ 000 = \log 10^3 = 3$

Rules for logs

Rule 1

$$\log (a \times b) = \log a + \log b$$

Try the following

$$100\ 000 = 1\ 000 \times 100$$
$$\log (100\ 000) = \log 1\ 000 + \log 100$$
$$\log 10^5 = \log 10^3 + \log 10^2$$
$$5 = 3 + 2$$

Rule 2

$$\log (a^n) = n \log a$$

Try the following

$$\log (5^3) = 3 \log 5$$
$$2.10 = 3 \times 0.699$$
$$= 2.10$$

Try the following

$10^6 = 1\,000\,000 = 10 \times 10 \times 10 \times 10 \times 10 \times 10$

$\log 10^6 = \log (10 \times 10 \times 10 \times 10 \times 10 \times 10)$

$\log 10^6 = \log 10 + \log 10 + \log 10 + \log 10 + \log 10 + \log 10 = 6 \log 10$

$6 = 1 + 1 + 1 + 1 + 1 + 1 = 6 \times 1$

Rule 3

$\log_x x^n = n$

This is because $\log_x$ is the inverse operation of x^n

So for example $\log 10^8 = 8$ since log represents $\log_{10}$.

Let's apply these rules to our original problem….. How to find the unknown power in an equation

$y = kx^n$ and I don't know n

Take log to the base 10 of each side of the equation

$\log y = \log (kx^n)$

Apply rule 1

$\log y = \log k + \log x^n$

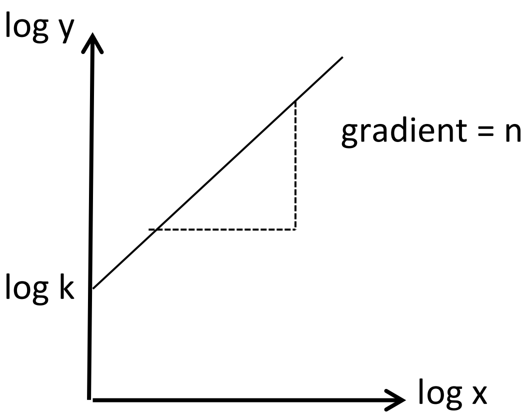
Apply rule 2

$\log y = \log k + n \log x$

Rearrange this new equation into the form Y = mX + c

$$\begin{array}{ccccc} \log y & = & n \; \log x & + & \log k \\ \downarrow & & \downarrow \; \downarrow & & \downarrow \\ Y & = & m \; X & + & C \end{array}$$

So if we plot log y against log x, the unknown power is now the gradient and the constant k can now be found from the y intercept, which is log k.



So we now have a method for determining an unknown power when processing data.

Practice Question

The relationship between the time period of a pendulum T (time for one oscillation) and L the length of the pendulum is given by

$$T = k L^n$$

Determine n from the following experimental data and determine k the constant of proportionality.

T/s	1.79	2.03	2.20	2.30	2.46
L/m	0.80	1.00	1.20	1.30	1.50

Firstly take logs of each side of the equation

$\log T = \log (kL^n)$

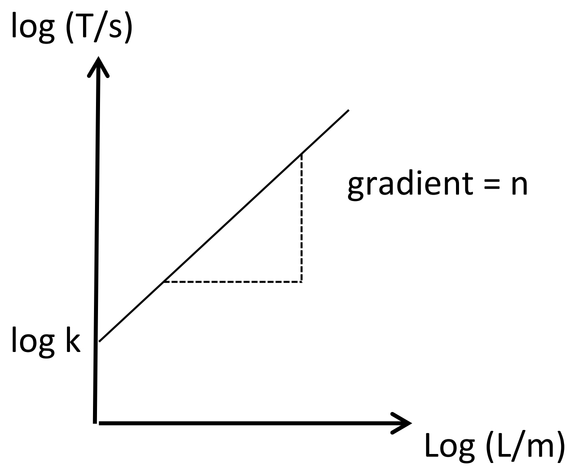
$\log T = \log k + \log L^n$

$\log T = \log k + n \log L$

$\log T = n \log L + \log k$

$Y = mX + c$

We can plot log T on the Y axis and log L on the X axis. The power n will be the gradient and log k will be the Y intercept.



The log of a number is a power and as such has no unit. However both T and L have units, so we write log of T as log (T/s) to indicate that the log itself has no units but T is measured in seconds. Similarly the log of L is written as log (L/m).

Now make a new table to record the log of T and log of L

log (T/s)	0.253	0.307	0.342	0.362	0.391
log (L/m)	-0.10	0.00	0.079	0.114	0.176

Note, the log of a number less than 1 is always negative, also log to the base 10 of 1 is 0.

If a graph is plotted and the gradient of the line calculated, n = ½ and log k = 0.307, so k = 2.03.

Log to the base e

Also known as the natural logarithm (ln). The base of the natural logarithm is the number e, which is equal to 2.718…

The natural logarithm of a number is the power to which e must be raised in order to equal that number.

The rules for logarithms apply to all no matter what the base.

Exponentials

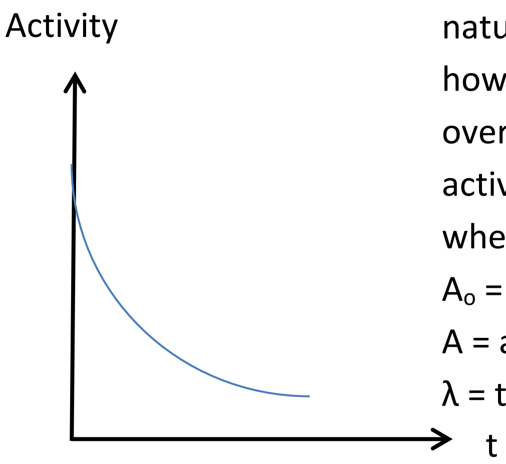
In science many variations of quantities, one with respect to another, are not in a linear fashion.

The change, growth or decrease, may occur rapidly.

Two examples of such are the exponential variation with time are radioactive decay in capacitor, charge and discharge.

The characteristic of an exponential decay is that it represents a quantity which decreases by the same fraction in successive equal time intervals.

Radioactive Decay



This is a relationship which you should have already encountered at GCSE. Radioactivity is a naturally occurring phenomena which describes how the radioactive decay of a substance reduces over a period of time. The relationship between activity A and the time t is written as $A = A_o e^{-\lambda t}$ where

A_o = activity at time = 0

A = activity at time = t

λ = the decay constant

If we recorded data of activity A over a period of time, the constant λ could be determined by taking log to the base e written as $\log_e$ or **ln** on a calculator.

$$A = A_o\, e^{-\lambda t}$$

$$\log_e A = \log_e (A_o\, e^{-\lambda t})$$

$$\log_e A = \log_e A_o + \log_e e^{-\lambda t}$$

$$\log_e A = \log_e A_o - \lambda t$$

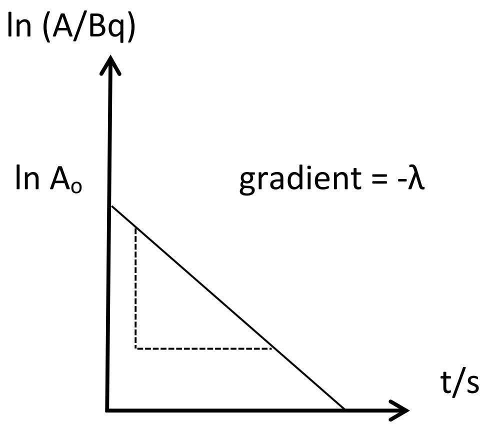
this is written as

$$\ln A = -\lambda t + \ln A_o$$

$$\downarrow \quad \downarrow\downarrow \quad \downarrow$$

$$y \;= m\, x + c$$

So by plotting log to the base e of activity against time we will get a straight line graph with a negative gradient of magnitude λ and a positive Y intercept of ln A_o.



We only use log to the base e (written ln) when dealing with relationships involving exponential decay or exponential growth.

The two examples in A level Physics are radioactivity and charging or discharging capacitors.