Polya Enumeration:

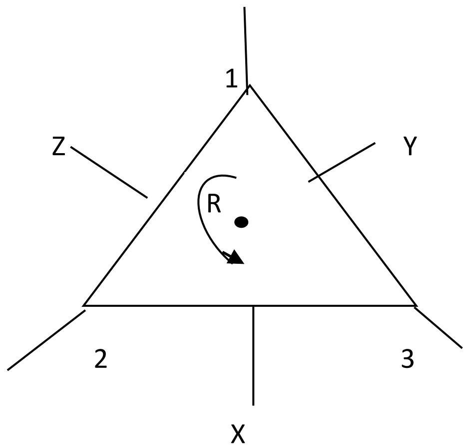
The following extends the material on Groups so that a powerful technique for enumeration can be covered.

Description of symmetry groups using the cycle index for regular polygons

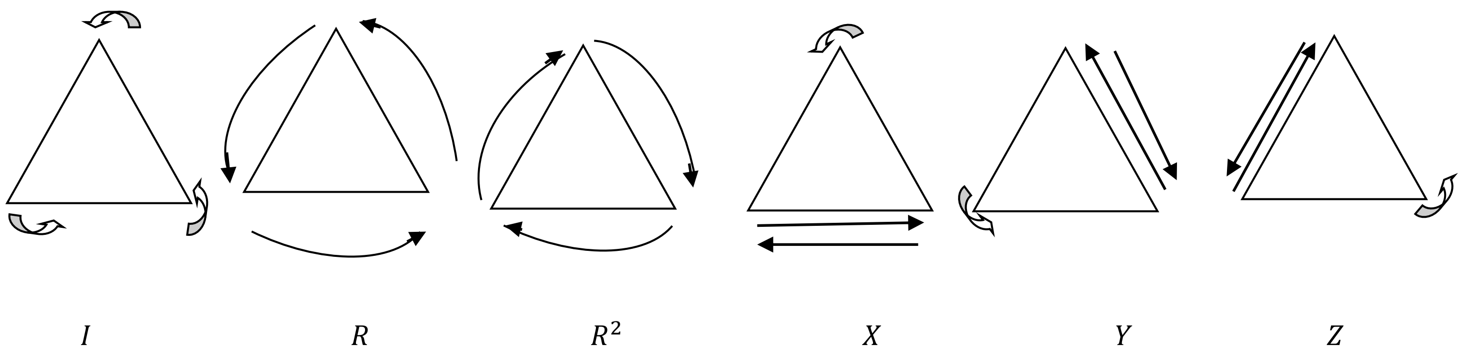
Very often different arrangements can be considered equivalent if we accept that the two arrangements are symmetrically the same, that is, one can be transformed into the other by an element of a symmetry group. For example, if the vertices of an equilateral triangle are coloured blue or green, then the arrangements BBG, BGB and GBB are in practice equivalent and there is only one way. An insight into why this is so will necessarily take into account the symmetry group of the equilateral triangle, and particularly how each element of the symmetry group will permute the three vertices.

Before we can outline how the mathematician Polya approached these types of problems, it is important to classify symmetries through how each vertex (or each element of the permutation group) behaves.

Consider the equilateral triangle. The symmetry group is the Dihedral group $D_3 = \left(\left\{I, R, R^2, X, Y, Z\right\}, \circ\right)$, where the elements are as shown in the diagram.



Each of these symmetries can be classified by the cycles for each vertex, that is, sometimes the vertex has a cycle of length 1, sometimes of length 2 and sometimes of length 3.



So the diagram for X shows one vertex in a cycle of cycle length 1 and the other two vertices in a cycle of length 2. We summarise this by the monomial $x_1 x_2$.

In general, the monomial $k{x_p}^r {x_q}^s$ summarises that there k different elements of the symmetry group, each of which has a r cycles of length p and s cycles of length q .

So for the vertices of the equilateral triangle we have:

symmetry	I	R	R^2	X	Y	Z
Classification of cycles	Three cycles of cycle length 1	One cycle of length 3	One cycle of length 3	One of length 1 and one of length 2	One of length 1 and one of length 2	One of length 1 and one of length 2
Vertex cycle monomial	x_1^3	x_3	x_3	$x_1 x_2$	$x_1 x_2$	$x_1 x_2$

Note: the subscript denotes the cycle length and the superscript denotes the number of cycles. When the superscript is 1, it is usually omitted.

The cycle index (or cycle indicator) of the group D$_3$ acting on Δ is defined as

$P_{D_3}(\Delta) = \frac{1}{|D_3|} \sum_{\sigma \in D_3} x(\sigma) = \frac{1}{6}\left(x_1^3 + 3x_1x_2 + 2x_3\right)$, where σ denotes each symmetry or permutation.

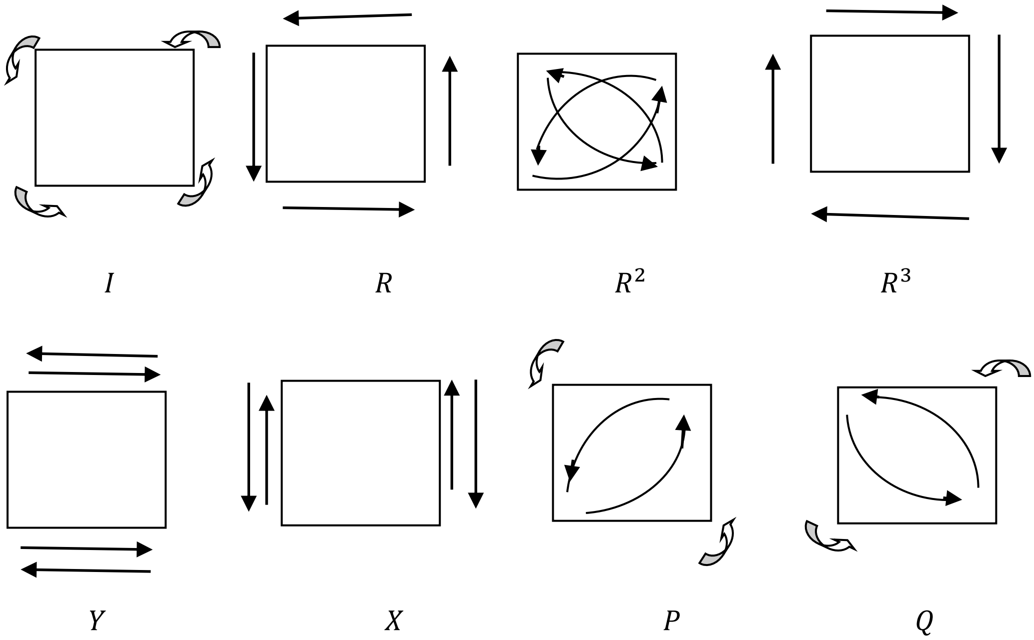
This can also be written as $P_{ET}(x_1, x_2, x_3) = \frac{1}{6}\left(x_1^3 + 3x_1x_2 + 2x_3\right)$ where *ET* denotes the symmetry group of the equilateral triangle and x_i is the dummy variable corresponding to the cycle of length i . This cycle index summarises how the vertices permute under the action of the symmetry group.

Note: it is important as a check to ensure that:

- the coefficients of the monomials sum to the order of the group, for example, 1+3+2 = 6
- for each monomial the sum of the products of the subscripts and superscripts is the same , for example, in $k{x_p}^r {x_q}^s$ we have the sum $pr + qs$. For example in x_1^3 and $3x_1 x_2$ we have sum = 3

Before moving on to look at the square, it is worth noting that in the above analysis we were looking at the permutations of the vertices. We could have looked at the permutations of the edges and in this case reached the same cycle index. In most cases, especially with polyhedrons, as we will see, the cycle indices for the vertices and edges will be different.

The similar analysis for the symmetry group of the square, called the Dihedral group *D*$_4$, is:



The cycle monomials (for the permutations of the vertices) are as given in the table below:

symmetry	I	R	R²	R³	Y	X	P	Q
Vertex cycle monomial	x_1^4	x_4	x_2^2	x_4	x_2^2	x_2^2	$x_1^2 x_2$	$x_1^2 x_2$

The cycle index is $P_{D_4}(\square) = \frac{1}{|D_4|}\sum_{\sigma \in D_4} x(\sigma) = \frac{1}{8}\left(x_1^4 + 2x_1^2 x_2 + 3x_2^2 + 2x_4\right)$

Or $P_S\left(x_1, x_2, x_4\right) = \frac{1}{8}\left(x_1^4 + 2x_1^2 x_2 + 3x_2^2 + 2x_4\right)$

Once again if we had looked at the permutations of the edges, we would have arrived at a similar cycle index, even though, each specific cycle monomial may be different. As noted before, in most cases, the cycle index for the vertices and edges will be different.

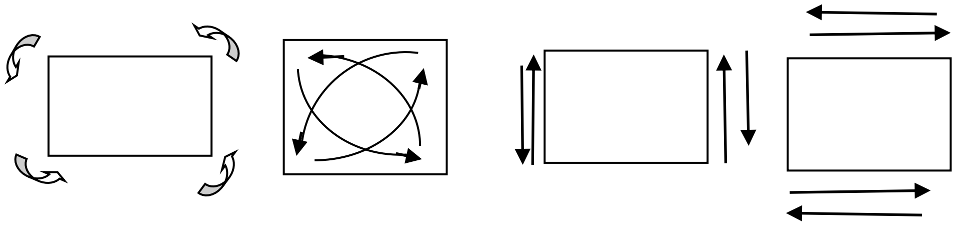
symmetry	I	R	R²	R³	Y	X	P	Q
Edge cycle monomial	x_1^4	x_4	x_2^2	x_4	$x_1^2 x_2$	$x_1^2 x_2$	x_2^2	x_2^2

<u>Worked Example</u>

Find the cycle index for the symmetry group of the rectangle (also called the Dihedral group D_2) acting on (i) the vertex permutations of the rectangle, and (ii) the edge permutations of the rectangle.

<u>Solution</u>

(i) The symmetries can be labelled R (rotation through 180°), X (reflection in horizontal axis) and Y (reflection in vertical axis). The diagrams below show the cycles pictorially



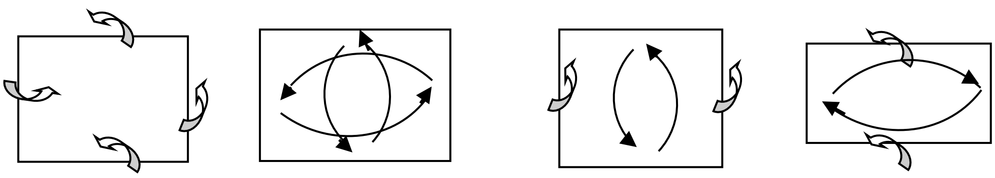
I *R* *X* *Y*

The cycle monomials are

symmetry	I	R	X	Y
Vertex cycle monomial	x_1^4	$x_2^{\ 2}$	x_2^2	x_2^2

So the vertex cycle index is $P_{R_{vertex}}(x_1, x_2) = \frac{1}{4}\left(x_1^{\ 4} + 3x_2^{\ 2}\right)$

Consider now the edges. The diagram below shows the cycles



I *R* *X* *Y*

The cycle monomials are

symmetry	I	R	X	Y
Edge cycle monomial	x_1^4	$x_2^{\ 2}$	$x_1^{\ 2}x_2$	$x_1^{\ 2}x_2$

So the edge cycle index is $P_R(x_1, x_2) = \frac{1}{4}\left(x_1^{\ 4} + x_2^{\ 2} + 2x_1x_2^{\ 2}\right)$

Exercise 1
1.1 Find the vertex cycle index for equilateral triangle under only the rotation symmetries.
1.2 Find the vertex cycle index for the square under only the rotation symmetries.
1.3 Find the edge cycle index for the square under only the rotation symmetries.
1.4 Find the vertex cycle index for the regular pentagon under only rotation symmetries.
1.5 Find the vertex cycle index for the regular pentagon under all its symmetries.

The formulae booklet gives the cycle index for a regular polygon with prime vertices and the cycle indices for the square, hexagon, octagon, nonagon, decagon and dodecagon. The first of these is worth looking at. Comparing it with the equilateral triangle and the answer to question 1.5 we have:

	Identity	Rotational symmetry	Bilateral symmetry
Equilateral triangle	3 cycles of cycle length 1	2 symmetries, each of which comprises cycles of cycle length 3	3 different symmetries, each of which comprises 1 cycle of length 1 and 1 cycle of length 2
Regular pentagon	5 cycles of cycle length 1	4 symmetries, each of which comprise cycles of cycle length 5	5 different symmetries, each of which comprises 1 cycle of length 1 and 2 cycles of length 2
Regular polygon with prime vertices	p cycles of cycle length 1	$(p-1)$ different symmetries, each of which comprises a cycle of cycle length p	p different symmetries, each of which comprises 1 cycle of cycle length 1 and $(p-1)/2$ cycles of cycle length 2

The rotational symmetry component arises because each element of the cyclic subgroup can generate the subgroup. The bilateral symmetry component arises because each line of symmetry through a vertex leaves $(p-1)/2$ pairs of remaining vertices. So, the general formula for a regular polygon with prime vertices is $\frac{1}{2p}\left(x_1^p + (p-1)x_p + px_1x_2^{\frac{(p-1)}{2}}\right)$.

Exercise 2

2.1 Consider the rotational symmetries of the regular hexagon and explain how the monomials x_6 , x_3^2 and x_2^3 arise.

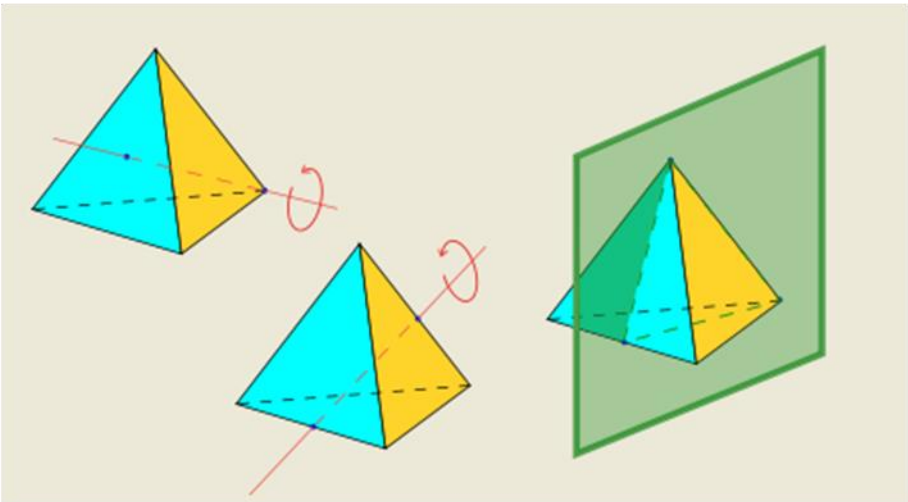
2.2 Consider the bilateral symmetries of the regular hexagon and explain how the monomials x_2^3 and $x_1^2 x_2^2$ arise.

2.3 Explain why the cycle index for the full symmetry group of the regular hexagon is

$$P_H\left(x_1, x_2, x_3, x_6\right) = \frac{1}{12}(x_1^6 + 4x_2^3 + 2x_3^2 + 2x_6 + 3x_1^2 x_2^2)$$

Description of symmetry groups using the cycle index for regular polyhedrons

Consider now the face permutation group of the regular tetrahedron. There are eight rotational symmetries about axes through vertices (+120° and -120° for each of the four axes). There are three rotational symmetries about axes through mid-points of opposite edges (180°), and six reflections about planes that include an edge and the mid-point of the opposite edge.



The face permutation monomials for these are respectively: $x_1 x_3$, x_2^2 and $x_1^2 x_2$

However, these 18 symmetries (17 + the identity) are not complete and do not form a group. [Note: the full permutation group of 4 things has 4! permutations and so order 24. By Lagrange's Theorem, we know it can't have a sub-group of size 18. In fact, combining some of these 18 would form 6 new composite permutations, which are called rotoreflections.]

If we restrict ourselves to the physically possible orientations of a tetrahedron, we can discount the reflections (and the rotoreflections) and obtain the cycle index $\frac{1}{12}\left(x_1^4 + 8x_1 x_3 + 3x_2^2\right)$. [Note: think of these 12 orientations and the other 12 arising from the mirror image of the initial tetrahedron.]

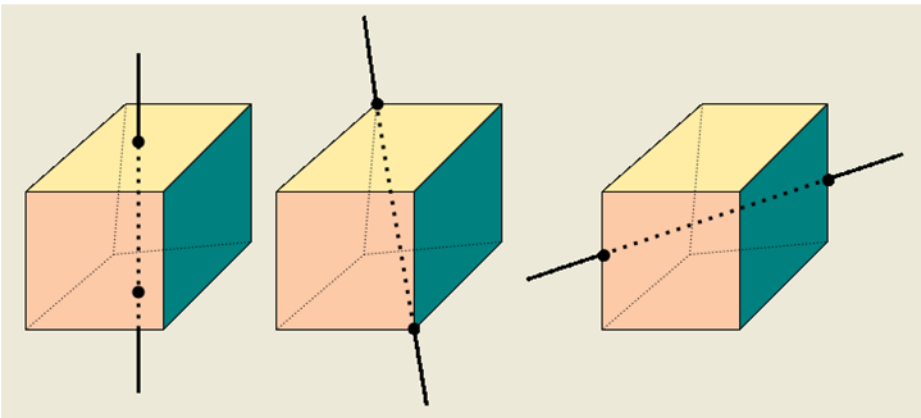
If, instead, we had looked at the vertex permutations, we would have arrived at the same cycle index. This arises as the tetrahedron is its own dual.

However, if we had examined the edge permutations, we would have arrived at x_3^2 and $x_1^2x_2^2$ respectively for the two rotations shown in the diagram above. The cycle index for the edge permutations is $\frac{1}{12}\left(x_1^6+8x_3^2+3x_1^2x_2^2\right)$.

Again, it is important as a check to ensure that:

- the coefficients of the monomials sum to the order of the group, for example, 1+8+3 = 12
- for each monomial the sum of the products of the subscripts and superscripts is the same , for example, in $kx_p{}^r x_q{}^s$ we have the sum $pr+qs$, so in $x_1{}^6$, $8x_3{}^2$ and $3x_1{}^2x_2{}^2$ we have sum = 6 = number of edges

Consider now the face permutation group of the cube. The cube has 48 symmetries when mirror images are included, but 24 when only physically possible orientations are counted. The rotations can occur about three different sets of axes.



In the first diagram, there are three such axes that result in 3 rotations of +90°, 3 rotations of 180° and 3 rotations of -90°, which have cycle monomials $x_1^2x_4$, $x_1^2x_2^2$ and $x_1^2x_4$ respectively.

In the second diagram, there are four such axes that result in 4 rotations of +120° and 4 rotations of -120°, each of which has cycle monomial x_3^2 .

In the third diagram, there are six such axes that result in 6 rotations of 180°, each of which has a cycle monomial x_2^3 .

So altogether the cycle index for the face permutations of the cube is

$$\frac{1}{24}\left(x_1^6+6x_1^2x_4+3x_1^2x_2^2+8x_3^2+6x_2^3\right).$$

The formulae booklet also provides the cycle indices for the octahedron and it can be seen the vertices and faces of the octahedron match the faces and vertices for the cube respectively. This is to be expected as the cube and octahedron are duals.

Exercise 3

3.1 Explain why the cycle index for the vertex permutations of the cube is

$$\frac{1}{24}\left(x_1^{\,8}+8x_1^{\,2}x_3^{\,2}+9x_2^{\,4}+6x_4^{\,2}\right)$$

Polya enumeration

While the cycle index helps to describe the symmetries of each shape, their main purpose is to help count the number of ways the vertices, edges or faces can be coloured.

For example, suppose we want to colour the corners of a square with 2 colours. All we have to do is replace each x_i by 2. That is, $\frac{1}{8}\left(2^4+2\left(2^3\right)+3\left(2^2\right)+2\left(2\right)\right)=\frac{1}{8}\left(16+16+12+4\right)=6$. The 6 ways are shown in the diagram where, for example, R and B indicate red and blue.

R R   R B
R R   R R

R R   R B
B B   B R

B B   B R
B B   B B

Furthermore, we could have asked how many ways was possible using n colours and replaced 2 by n . The answer would have been $\frac{1}{8}\left(n^4+2n^3+3n^2+2n\right)$

Worked Example

A Christmas tree star is based on a five pointed regular star with a light at each of the points. How many ways are there of using three different coloured light bulbs?

Solution

The five lights will form a regular pentagon, which has a vertex cycle index of $\frac{1}{5}\left(x_1^{\,5}+4x_5+5x_1x_2^{\,2}\right)$.

So replacing x_i by 3, we obtain $\frac{1}{5}\left(3^5+4\left(3\right)+5(3)(3^2)\right)=78$. So there are 78 ways.

Exercise 4

4.1 Show that there are 57 ways of colouring the faces of a cube with three colours.

4.2 Show that the number of ways of colouring the faces of a tetrahedron using n colours is $\frac{1}{12}\left(n^4+11n^2\right)$

4.3 How many ways are there of colouring the faces of an octahedron with 2 colours?

The Polya Enumeration Theorem

In 1937 the Hungarian mathematician Polya generalised earlier work by showing that the x_i can be replaced by algebraic expressions, such as $\left(R^i + B^i\right)$, to get detailed information about the different ways of colouring.

For example, in the case of colouring the corners of a square using red and blue, see earlier, we have

$$\frac{1}{8}\left(x_1^4 + 2x_1^2 x_2 + 3x_2^2 + 2x_4\right) \rightarrow \frac{1}{8}\left(\left(R+B\right)^4 + 2\left(R+B\right)^2\left(R^2+B^2\right) + 3\left(R^2+B^2\right)^2 + 2\left(R^4+B^4\right)\right)$$

$$= \frac{1}{8}\left(8R^4 + 8R^3B^1 + 16R^2B^2 + 8R^1B^3 + 8B^4\right) = R^4 + R^3B^1 + 2R^2B^2 + R^1B^3 + B^4$$

It is simple to see where this algebraic expression corresponds to the six arrangements shown opposite.

R R	R B		
R R	R R		
		R R	R B
		B B	B R
B B	B R		
B B	B B		

That is, one arrangement of 4 red corners, another of 4 blue corners, one of 1 red and 3 blue, one of 3 red and 1 blue, and finally two arrangements of 2 red and 2 blue.

Worked example

Consider the three vertices of an equilateral triangle. What are the possible arrangements of colouring the vertices using red, blue and green?

Solution

The cycle index is $P_{ET}\left(x_1, x_2, x_3\right) = \frac{1}{6}\left(x_1^3 + 3x_1x_2 + 2x_3\right)$. So substituting in the appropriate terms we have

$$\begin{aligned}
&\frac{1}{6}\left(\left(R+B+G\right)^3 + 3(R+B+G)(R^2+B^2+G^2) + 2(R^3+B^3+G^3)\right) \\
&= \frac{1}{6}((R^3+B^3+G^3+3R^2B+3RB^2+3R^2G+3RG^2+3B^2G+3BG^2+6RBG) \\
&\qquad +3(R^3+B^3+G^3+RB^2+RG^2+BR^2+BG^2+GR^2+GB^2) \\
&\qquad\quad +2(R^3+B^3+G^3)) \\
&= \frac{1}{6}(6R^3+6B^3+6G^3+6R^2B+6RB^2+6R^2G+6RG^2+6B^2G+6BG^2+6RBG) \\
&= R^3+B^3+G^3+R^2B+RB^2+R^2G+RG^2+B^2G+BG^2+RBG
\end{aligned}$$

So we have one arrangement with 3 red vertices, one with 3 blue vertices, one with 3 green vertices, one with 2 red and 1 blue, etc, and finally one arrangement with 1 red, 1 blue and 1 green.

<u>Exercise 5</u>

5.1 Using two colours, red and blue, what are the different arrangements of colouring the vertices of a regular pentagon?

5.2 Using two colours, red and blue, what are the different arrangements of colouring the vertices of a regular hexagon?

5.3 Using two colours, red and blue, what are the different arrangements of colouring the faces of a regular cube?

5.4 Using three colours, red, blue and green, what are the different arrangements of colouring the faces of a regular tetrahedron?

5.5 The full edge permutation cycle index for the tetrahedron $\frac{1}{24}\left(x_1^6+9x_1^2x_2^2+8x_3^2+6x_2x_4\right)$, that is, using the 12 physically possible transformations and the 12 reflections and rotoreflections.

Comparing the different graphs with 4 vertices, that is, from N_4 to K_4 , with the colouring of the edges of a regular tetrahedron either black or white, show that there are eleven different graphs with 4 vertices, that is, one with no edges (N_4), one with 1 edge, two with 2 edges, three with 3 edges, two with 4 edges, one with 5 edges and one with 6 edges (K_4)

Answers to exercises

Exercise 1

1.1 $P_{ET}(x_1, x_2) = \frac{1}{3}(x_1^3 + 2x_3)$

1.2 $P_S(x_1, x_2, x_4) = \frac{1}{4}(x_1^4 + x_2^2 + 2x_4)$

1.3 $P_S(x_1, x_2, x_4) = \frac{1}{4}(x_1^4 + x_2^2 + 2x_4)$

1.4 $P_P(x_1, x_5) = \frac{1}{5}(x_1^5 + 4x_5)$

1.5 $P_P(x_1, x_2, x_5,) = \frac{1}{10}(x_1^5 + 4x_5 + 5x_1x_2^2)$

Exercise 2

2.1 The rotation through 0° or identity has monomial x_1^6 . The rotations (clockwise and anti-clockwise) through 60° have monomial x_6 . The rotations (clockwise and anti-clockwise) through 120° have monomial x_3^2 . The rotation through 180° has monomial x_2^3 .

2.2 The bilateral symmetries about the axes through the opposite vertices have monomials $x_1^2x_2^2$. The bilateral symmetries about the axes through the midpoints of opposite edges have monomials x_2^3.

2.3 There is one identity (x_1^6), there are two rotations through 60° (x_6), two rotations through 120° (x_3^2), one rotation through 180° (x_2^3), three reflections about the axes through opposite vertices ($x_1^2x_2^2$) and three reflections about the axes through the midpoints of the opposite edges (x_2^3).

Exercise 3

3.1 In the first diagram, there are three such axes that result in 3 rotations of +90°, 3 rotations of 180° and 3 rotations of -90°, which have cycle monomials x_4^2 , x_2^4 and x_4^2 respectively. In the second diagram, there are four such axes that result in 4 rotations of +120° and 4 rotations of -120°, each of which has cycle monomial $x_1^2x_3^2$. In the third diagram, there are six such axes that result in 6 rotations of 180°, each of which has a cycle monomial x_2^4 . And, of course, there is also the identity x_1^8

Exercise 4

4.3 23 ways

Exercise 5

5.1 One way for 5 red, one for 4 red and 1 blue, two for 3 red and 2 blue, two for 2 red and 3 blue, one for 1 red and 4 blue and one for 5 blue.

In the case of 3 red and 2 blue, the blues can be adjacent or separated by one vertex.

Similarly, in the case of 2 red and 3 blue, the reds can be adjacent of separated by one vertex.

5.2 One way for 6 red, one for 5 red and 1 blue, three for 4 red and 2 blue, three for 3 red and 3 blue, three for 2 red and 4 blue, one for 1 red and 5 blue and one for 6 blue.

In the case of 4 red and 2 blue, the blues can be in positions 1 and 2, 1 and 3, or 1 and 4.

Similarly, the 2 reds can be in the same positions with 4 blue and 2 red.

In the case of 3 red and 3 blue, the reds can be in positions 1, 2 and 3, or in positions 1, 2 and 4, or in positions 1, 3 and 5

5.3 One way of arranging 6 red, one way of 5 red and 1 blue, two ways of 4 red and 2 blue, two ways of 3 red and 3 blue, two ways of 2 red and 4 blue, one way of 1 red and 5 blue, and one way of arranging 6 blue.

The two ways for 4 red and 2 blue are the 2 blue arranged as either adjacent faces or opposite faces. Similarly with 2 red and 4 blue, the 2 reds are either adjacent or opposite.

The two ways for 3 red and 3 blue are the reds arranged as either the 3 faces meeting at a vertex or the three faces arranged in a 'strip', that is face a adjacent to face b which is in turn adjacent to face c, where face c is not adjacent to face a but opposite to face a.

5.4 One of the following: 4 red, 4 blue, 4 green, 3 red and 1 blue, 2 red and 2 blue, 1 red and 3 blue, 3 red and 1 green, 2 red and 2 green, 1 red and 3 green, 3 blue and 1 green, 2 blue and 2 green, 1 blue and 3 green, 2 red and 1 blue and 1 green, 1 red and 2 blue and 1 green, and finally 1 red and 1 blue and 2 green.

5.5.

$$\frac{1}{24}\left(x_1^6+9x_1^2x_2^2+8x_3^2+6x_2x_4\right)\to\frac{1}{24}\left(\left(B+W\right)^6+9\left(B+W\right)^2\left(B^2+W^2\right)^2+8\left(B^3+W^3\right)^2+6\left(B^2+W^2\right)\left(B^4+W^4\right)\right)$$

$$=W^6+W^5B+2W^4B^2+3W^3B^3+2W^2B^4+WB^5+B^6$$